MAKING SENSE OF FRACTIONS GIVEN WITH DIFFERENT SEMIOTIC REPRESENTATIONS

Frode Rønning

Sør-Trøndelag University College

N-7004 Trondheim

Norway

This paper is based on observations of a group of 20 pupils in grade four in a Norwegian primary school. The pupils are presented with a task involving fractions. In the task the pupils are asked to judge the relative size of some simple fractions and also to identify equivalent fractions. The fractions in the task are linked to a context about milk boxes with different volume and the task also involves conversions between measuring units. These pupils are at a very early stage in their learning of fractions and my interest is mainly in inquiring into how they make sense of the situations they are exposed to, with special emphasis on semiotic representations.

Keywords: Fractions, semiotic representations, mediating artefacts.

INTRODUCTION

In a previous paper for CERME (Rønning, 2010) I have reported on a study of grade 4 children involved in the practical task of measuring out 15 dl of milk from boxes containing a quarter of a litre. The boxes were labelled 1/4 liter[1] and in that paper I discussed how the children interpreted the sign 1/4 and to what extent the interpretation had any effect on completing the measuring task. I showed that the presence of a measuring beaker as a mediating artefact to a large extent made it redundant to actually make sense of the sign 1/4. In this paper I report on a study of the exact same pupils, grouped in the same groups as before, working on a task that is mathematically similar to the measuring task reported on before but in terms of the representations used it is quite different. Here the children are presented with a task, given as a text accompanied by pictures, which they are asked to discuss and solve. My main research question is how the children make sense of the fractions given with different representations. I am in particular interested in how the children argue about the relative size of the fractions, how they argue about equivalent fractions, and how they handle fractions larger than 1. I will make connections to the situation described in (Rønning, 2010) and compare this to the situation described here.

THEORETICAL FRAMEWORK

In this study the notion of a sign is central. According to Steinbring a sign typically has two functions, a semiotic function; “something that stands for something else”, and an epistemological function, indicating “possibilities with which the signs are endowed as means of knowing the objects of knowledge” (Steinbring, 2006, p. 134). What is special for mathematics, in contrast to other subjects in school, is that all the objects of study are abstract and that they can only be accessed using signs and semiotic representations (Duval, 2006, p. 107). Despite the abstract nature of the mathematical objects, mathematics is used as a tool to describe and make predictions about real life situations. A sign can therefore refer both to a mathematical concept as well as to a real life situation. In this study this dual nature can be exemplified through the sign 1/4, used as a representation for the mathematical concept "the fraction one over four", as well as for an actual quantity of milk, contained in a real or imaginary milk box. Using Steinbring's construct *The epistemological triangle* (2006, p. 135) the amount of milk in one box is the object or reference context and the concept is the idea of the fraction 1/4.

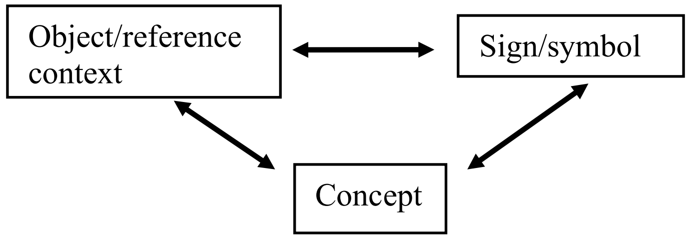

Figure 1: The epistemological triangle

In most cases a mathematical concept can have multiple representations with different characteristics. According to Peirce a sign can be an *icon,* which stands for its object by likeness; an *index*, which stands for its object by some real connection with it; or a *symbol,* which is only connected to the object it represents by habit or by convention (Peirce, 1998, pp. 13-17, 272-275). Successful learning of mathematics is often linked to the ability to switch between different representations of the same mathematical object. Being able to do this and keeping the connection to the same object is, according to Duval (2006), one of the most important obstacles to learning mathematics. In my paper iconic (depictive) and symbolic (descriptive) representations will play the main role. Depictive representations possess inherent structural features making it possible to extract relational information but they do not contain symbols for these relations. Descriptive representations also contain information about relations but to extract this information it is necessary to know the conventions embedded in the symbols (Schnotz & Bannert, 2003, p. 143).

Making sense of conventional representations can be thought of as creating strong links between the corners of the epistemological triangle. In the process of creating such links the learner often uses *hedge words* (Lakoff, 1973; Rowland, 2000) as an indicator of uncertainty, lack of a precise language, or as a search for approval from e.g. a teacher. Rowland groups hedge words into two main categories, *shields* and *approximators* (2000, pp. 60-61). In brief, shields indicate a vagueness on behalf of the speaker, that he or she does not guarantee the truth of the proposition to follow, whereas approximators indicate a vagueness in the proposition (that the speaker is aware of).

Much research has been done on learning with multiple representations and it has been claimed that using multiple representations will enhance learning. Susan

Ainsworth has identified three main functions of using multiple representations; complementary roles, to constrain interpretation, and to construct deeper understanding (Ainsworth, 1999, p. 134, 2006, p. 187). The first function is to use representations containing complementary information or supporting complementary cognitive processes. The second function is to use one representation to constrain learners' interpretation of another representation, e.g. to support the interpretation of an abstract representation. The third function has to do with encouraging learners to construct a deeper understanding of a situation.

In school mathematics fractions are in most respects taken to be equivalent to positive rational numbers. Many authors have described subconstructs of rational numbers and the list of subconstructs varies among different authors. Behr, Lesh, Post, and Silver (1983) claim that one can identify at least six different subconstructs: a part-to-whole comparison, a decimal, a ratio, an indicated division, an operator, and a measure of continuous or discrete quantities. According to Behr et al. the part-whole subconstruct is fundamental to all later interpretations and these authors also suggest a reconceptualisation of this subconstruct into what they denote as *the fractional measure subconstruct*. Previous studies about the learning of fractions have often been of a quantitative nature and focussed on the various subconstructs of rational numbers (see e.g. Behr et al., 1983). Recently there seems to have developed a stronger interest in looking at the learning of fractions through transformations between multiple forms of representation (see e.g. Ebbelind, Roos, & Nilsson, 2012). This study shares much of its theoretical foundation with my study.

METHOD

The 20 pupils in the class were grouped into four groups, each consisting of five pupils. Each group left the regular teaching in the class and came to a nearby room where I was waiting for them. I sat together with the children around a table and each child received a sheet of paper with the task written on it. They had not seen the task before or been given any information about what they were going to work on together with me. They could draw and write on the task sheet as well as on blank sheets that were available on the table. No concrete material was available. The episode was recorded by a video camera standing in a fixed position on a tripod. Each group of pupils got approximately 30 minutes for the task. The data for the analysis consist of the video recordings as well as the sheets of paper that the children used to write and draw on. Since I was actively taking part in the conversation I did not have the possibility to shift the position of the camera. It is directed towards the table and since the groups are so small it is to some extent possible to discern from the video the actual process when the children make drawings and link this to what was actually said at the same time.

To analyse the data I looked at the recordings and made summaries from the activities in each group, looking in particular for episodes that I considered important in relation to my research question. These episodes were transcribed, first in a style close to the spoken dialect and later to standard written Norwegian. For the purpose of this paper, parts of the transcriptions have been translated into English. Each section of utterances are given an internal numbering. All the children have been given English pseudonyms. The same child carries the same pseudonym in this article as in (Rønning, 2010). My analysis of the data is based on the pupils' utterances and their writings and drawings during the process, and through an interpretative process I will make some statements about the pupils' sense making of fractions. As an analytic tool I rely in particular on elements from semiotic theory and multimodal representations as well as theory about subconstructs of rational numbers.

In the task the pupils were presented with drawings of four different situations, each described by a figure as shown in Figure 2.

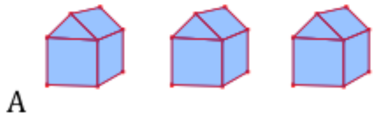

Figure 2: Situation from the pupils' task

In the text it was explained that the pictures illustrated milk boxes, blue and red. It was explained that each blue box contained 1/3 litre of milk, and each red box 1/4 litre[2] of milk. Situation A showed three blue boxes, situation B four blue boxes, situation C four red boxes, and situation D three red boxes. The questions in the task are reproduced below.

- Which box, red or blue, contains most milk?
- Which situation, A, B, C or D, represents the largest quantity of milk?
- And which situation represents the smallest quantity of milk?
- Are there any situations with the same amount of milk?
- How many decilitres of milk are there in situation D?
- You need 15 decilitres of milk and you have boxes containing 1/4 litre, hence red boxes. How many boxes do you need?

COMPARING FRACTIONS

The question of finding out which box, red or blue, contains most milk immediately turns into a question of comparing the fractions 1/3 and ¼, where the context with the milk boxes plays no role. The children introduce a new context where the fractions are represented by rectangular shaped figures, divided into parts in different ways. To illustrate 1/3 Fran draws a rectangle, divides it into three parts. Then she divides one of the three parts into two and shades this part to illustrate 1/4. Chris has a similar illustration of 1/4 as showed in Figure 3. An excerpt from a discussion in one of the groups is presented below.

Figure 3: Chris' illustration of 1/4

1.1 Fran: Because, when it is, in a way, a third, then it is divided into slightly larger pieces, but when it is a fourth, it is smaller. So there is more space in a third.

1.2 Chloe: If they get thirds, they get more, if they get fourths, they get less.

1.3 Chris: The larger the number below, the smaller is the actual part.

1.4 Frode: Yes, the smaller is the actual part.

1.5 Chris: It is almost like this. Then it is almost as if we split this one [points to the shaded part of Figure 3], making it even smaller.

The children create iconic signs (Peirce, 1998) to link to a reference context having to do with dividing a whole into pieces. The numerator (“the number below”, #1.2) is linked to the number of pieces and the size of each piece is linked to the size of the fraction in an inverse way. This is most clearly expressed by Chris when he states, “the larger the number below, the smaller is the actual part” (#1.3). When Chris links his statement about the numbers in the fraction (#1.3) to the illustration he is using hedge words, “almost like this” and “almost as if” (#1.5). I interpret these words to be *rounder approximators* whose effect is to “modify (as opposed to comment on) the proposition” (Rowland, 2000, p. 60). In his statement (#1.3) Chris shows no indication of vagueness, and therefore I interpret his statement (#1.5) not to be taken as an exact representation of a smaller fraction but as an approximation.

The signs 1/3 and 1/4 can be placed in an epistemological triangle dealing with the concept volume of one box where the blue and red boxes are placed in the reference context corner. Then the connection between sign and reference context is just a “communicative agreement” (Steinbring, 1998, p. 173). In order to reason about the relative size of the fractions it is necessary to introduce a new sign, the partitioned rectangles, that can function as a reference context for the signs 1/3 and 1/4. Then there is a structural reference between the sign and the reference context, “the connection between symbol and referent is indirectly mediated by syntactical and logical structures on the symbol level and the referent level” (Steinbring, 1998, p. 179). (See Figure 4. The sign > was not used in the interaction with the children.)

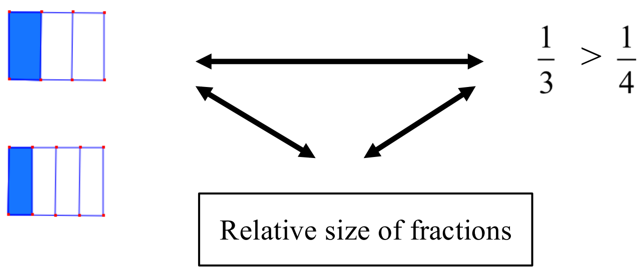

Figure 4: Comparing fractions

In the dialogue above Fran said that in 1/3 “it is divided into slightly larger pieces” (#1.1). In order to determine whether there are situations containing the same amount of milk a need to quantify the difference arises, *how much* larger is a blue box than a red box? Jessica and Ellie have suggested that situation A has the same amount of milk as situation C and in the dialogue below they justify their argument.

2.1	Jessica:	The blue one is in a way one more, I nearly said.
2.2	Ellie:	Then it is the same amount such that it is one more.
2.3	Jessica:	Yes. And then I think that the blues are almost one more than the reds.
2.4	Frode:	Yes …
2.5	Jessica:	One box more, yes there are three there [points to A] … and then you can in a way draw one more.

They agree that each blue box contains one third, and that the only possible solution, if any at all, is A and C.

2.6	Ellie:	But it is, the blue ones there [points to B], compared to the red ones, [points to C] there are five, and there are four, so it is like one more.
2.7	Frode:	OK.

The pupils have previously agreed that 1/3 is larger than 1/4 but how much larger? The answer given to this is that it is "one more" (#2.1). Also here the hedge words are interesting, e.g. when Jessica says that the blue box is "in a way one more" (#2.1) and "the blues are almost one more than the reds" (#2.3). This argument gives as a result that four red boxes are equal to three blue boxes. In this discussion I interpret the hedge words to be of the type that Rowland refers to as *adaptor approximators* (2000, pp. 60-61). Jessica attaches vagueness to the proposition (one more), not in the sense of a rounder (approximately one more) but in the sense that she does not have good way of expressing exactly how much larger 1/3 is than 1/4, so it is "in a way one more" . One may claim that Jessica, knowing that 4 is one more than 3 and that 1/3 > 1/4, the latter relation has a certain degree of "one more-ness" to it, albeit in a reciprocal way. The relation belongs, to a certain degree, to a category consisting of objects {*a, b*} where *a* is one more than *b* (Lakoff, 1973).

The argument is developed further into concluding that four blue boxes correspond to five red boxes, when Ellie says that "the blue ones there, compared to the red ones, there are five and there are four, so it is like one more" (#2.6). It is not clear what is meant by *one more* but I interpret Ellie's utterance to mean that four blue boxes correspond to five red boxes. Her reasoning works in the particular cases given by situation B and situation C but not in general. If v_r denotes the volume of one red box and v_b denotes the volume of one blue box, and n_r and n_b denote the number of red and blue boxes, respectively, then the total volume will be the same if $n_r \cdot v_r = n_b \cdot v_b$, or $n_r = \frac{v_b}{v_r} \cdot n_b$. Hence, the relation is a multiplicative one, whereas the pupils suggest that an additive relation, $n_r = n_b + 1$, will give the same volume.

LACK OF FLEXIBILITY IN THE REPRESENTATIONS

In the second question of the task the pupils were asked to identify the situation containing the largest amount of milk. Prior to this it had been established that the blue box contained the largest quantity of milk and the pupils quickly suggest that situation B has the largest amount of milk. Fran starts to justify this suggestion but soon she begins to object to her own suggestion.

3.1	Fran:	Because there [in B] there are many blue ones. The more blue ones, the … But really, it is not possible because the blue one is one third and this one is four thirds, so it really is not possible.
3.2	Frode:	Not possible. Why not?
3.3	Fran:	Because then there is one too many. Is it.
3.4	Chris:	That is why A is the largest. There there are three thirds.
3.5	Frode:	In A, yes.
3.6	Fran:	Because it is not possible with four thirds. That is not possible. If it had been fourths, then it had been possible.
3.7	Chloe:	There is only room for three in one litre. If it had been written four it would be possible. That is why it is possible in C.
3.8	Fran:	It is not possible to take one more box than what in a way is there.
3.9	Chloe:	So it is possible with one fourth but not with one third.
3.10	Fran:	So really, it is A.

There seems to develop a reluctance to accept situation B in this context, when Fran explicitly says “four thirds, so really it is not possible” (#3.1). Looking at the previous attempts of explaining fractions (see e.g. Figure 3) this reluctance is understandable. The fraction is compared to a fixed quantity, the unit, in this case drawn like an almost rectangular figure. The concept of fraction is taken as the part-whole subconstruct (Behr et al., 1983), meaning that there is a certain number of parts, n, and one of those is “one nth”. And it does not make sense to take $n + 1$ because the unit (the whole) does not contain more than n parts. “It is not possible to take one more box than what in a way is there”, as Fran puts it (#3.8). Or, “there is only room for three in one litre” (Chloe in #3.7). Here 1 litre is the limit, the whole. This shows the limitations involved when looking at fractions only as parts of a whole and how the representation given by partitioning a given unit is not sufficiently flexible.

To make sense of the symbolic representation $\frac{1}{n}$ all the pupils use a representation of the type shown in Figure 3, i.e. a rectangular shape partitioned into n stripes. This is an example of a depictive (iconic) representation that enables the pupils to achieve insight into the descriptive (symbolic) representation. In this way the multiple representations can be said to have a constructive function (Ainsworth, 1999, 2006). However, the iconic representation also has a constraining function, and in this case a constraining function that goes too far. Ainsworth describes the constraining function as being of help in the meaning making stating that it entails using one representation “to constrain possible (mis)interpretations in the use of another” (1999, p. 134). In the excerpt above one can see that the constraining is too strong in the sense that it does not allow for the sign “four thirds”. Situation B is a depiction of four blue boxes and this is expressed by Fran as being “four thirds”. This sign is never expressed in symbolic form. Chris writes $\frac{3}{3}$ next to the three blue boxes in Situation A and in

Situation B he draws a curve around three of the boxes and another curve around the fourth box, and says that “A is the largest” (#3.4).

A multiple representation for Situation B could be as shown in Figure 5.

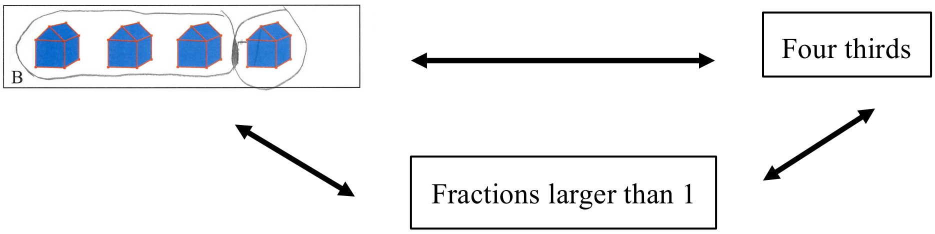

Figure 5: Reference context for fractions larger than one

Although the pupils use the verbal sign four thirds to represent Situation B, this is not accepted as a solution in this particular context, which we at an early stage in the conversation had agreed was about fractions. The epistemological triangle shown in Figure 5 for fractions larger than 1 is not established in the given situation. The pupils link the sign to the reference context but they cannot link this to a concept of fractions. Earlier they have linked the sign for the fraction (spoken or written) to the partitioned rectangle. The partitioned rectangle does not function as a link between the sign and the reference context in Figure 5 because the partitioned rectangle does not make sense in the case of four boxes of 1/3 each. Therefore there is a lack of flexibility in the representation, since one box corresponds to one stripe in the rectangle and there are only three stripes, so “it is not possible to take one more box than what in a way is there”, as Fran puts it (#3.8). Therefore, “really it is A [that has the largest amount of milk]” (#3.10).

In one of the other groups identifying the situation with the largest amount of milk presented no difficulties. Jessica read the question “Which situation, A, B, C, or D, contains the largest amount of milk?” and she immediately answers that it is B. This is repeated by Ellie and one of the other pupils. When asked why this is the case, Ellie says “because it is the largest number…“ . Ellie is interrupted by Jessica who gives the following explanation.

Jessica: Besides, you can see that it is most, and the reds are as many as the blues, so then you think, and here you said that blue was most, and it has to be blue which is most of red and blue there also.

Jessica accompanies her argumentation by pointing with her pencil to the task sheet and by combining what she says with the pointing that can be observed on the video I interpret her statement in the following way. She first verifies that the number of red boxes in C is the same as the number of blue boxes in B. Then she refers back to what we already had agreed on, that one blue box is more than one red box. Then she concludes that it must be the situation with four blue boxes (B) that contains the largest amount. I take it that she has tacitly assumed that none of the situations with three boxes could be a candidate for the largest amount.

Jessica's argument here does not need the concept of fractions larger than 1 so she does not face the same problems as the group with Fran, Chris and Chloe did. Jessica builds on previous knowledge that one blue box is larger than one red box ($v_b > v_r$) and for this the representations in Figure 3 can be used. Furthermore, by counting, it is clear that the number of blue boxes in B is equal to the number of red boxes in C ($n_b = n_r = n$). From this she concludes that $n\ v_b > n\ v_r$.

DISCUSSION

In this paper I have showed how representations that resonate strongly with the aspect of fraction as a part-whole relationship can be limiting and restrictive, in particular when it comes to dealing with fractions larger than 1. In a context, which in itself would be expected to be a familiar one (boxes of milk), one situation consisting of 4/3 litre was dismissed as being "not possible" and my interpretation of this is that because the task has to do with fractions, the pupils cannot accept four thirds, whereas they are happy to accept four fourths. It would have been of interest to pursue the statement "not possible" further by asking "what is not possible?" Instead I only ask "why it is not possible", assuming that we all have the same understanding of what "it" is. This can however, not be taken for granted.

Although mathematically the tasks for the pupils presented in this paper are quite similar to the ones discussed in (Rønning, 2010), there is one important difference. In the situation described in the previous paper the pupils had measuring devices available that functioned as mediating artefacts between the signs and the reference context. The presence of the mediating artefacts strongly reduced the need of making sense of the signs. Therefore some of the pupils read 1/4 as "one comma four[3]", and says "two comma eight" after having poured two 1/4 litre boxes of milk into the measuring beaker. One of the pupils saying "one comma four" is corrected by one of the other pupils who says that the sign / is not a comma but a slash (Rønning, 2010, p. 1018). The first pupil readily admits that she doesn't have a clue to what "one slash four litres" means but nevertheless the pupils in the group have no problems finishing the measuring task because they are guided by the measuring beaker where the scale has the function of an indexical sign (Peirce, 1998).

In the situation presented in this paper the depictions of the milk boxes in the task were not helpful in making sense of the fraction signs. Therefore the pupils created other signs that they could use but these signs turned out not to be sufficiently flexible to deal with the situations, in particular not in cases of fractions larger than 1.

REFERENCES

Ainsworth, S. (1999). The functions of multiple representations. *Computers & Education, 33,* 131-152.

Ainsworth, S. (2006). DeFT. A conceptual framework for considering learning with multiple representations. *Learning and Instruction, 16,* 183-198.

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 91-126). Orlando, FL: Academic Press.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61,* 103-131.

Ebbelind, A., Roos, H., & Nilsson, P. (2012). Learning fractions – transformations between representations from a social semiotic perspective of modality. In G. H. Gunnarsdóttir, F. Hreinsdóttir, G. Pálsdóttir, M. Hannula, M. Hannula-Sormunen, E. Jablonka, … K. Wæge (Eds.), *Proceedings of NORMA 11. The Sixth Nordic Conference on Mathematics Education in Reykjavík, May 11.-14. 2011* (pp. 217-226). Reykjavik: University of Iceland Press.

Lakoff, G. (1973). Hedges: A study in meaning criteria and the logic of fuzzy concepts. *Journal of Philosophical Logic, 2,* 458-508.

Peirce, C. S. (1998). *The essential Peirce. Selected philosophical writings. Vol. 2 (1893-1913).* (Edited by the Peirce Edition Project). Bloomington, IN: Indiana University Press.

Rowland, T. (2000). *The pragmatics of mathematics education. Vagueness of mathematical discourse.* London: Falmer Press.

Rønning, F. (2010). Tensions between an everyday solution and a school solution to a measuring problem. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.)*, Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education.* January 28th - February 1st 2009, Lyon (France) (pp. 1013-1022). Lyon: INRP.

Schnotz, W., & Bannert, M. (2003). Construction and interference in learning from multiple representation. *Learning and Instruction, 13,* 141-156.

Steinbring, H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematics Teacher Education, 1,* 157-189.

Steinbring, H. (2006). What makes a sign a mathematical sign? An epistemological perspective on mathematical interaction. *Educational Studies in Mathematics, 61,* 133-162.

---

[1] Norwegian spelling of litre.

[2] In the text presented to the pupils the signs $\frac{1}{4}$ and $\frac{1}{3}$ were used.

[3] In Norwegian a comma is used for the decimal point so "one comma four" means 1.4.