THE PRODUCTIVE ROLE OF INTERACTION: STUDENTS' ALGEBRAIC THINKING IN WHOLE GROUP DISCUSSION

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We report results from a design research of six lessons in a mathematics classroom with students aged 15 to 16. It has been examined the emergence of relationships between the generation of mathematical learning opportunities and the interaction in whole group. Together with the theoretical links between learning and learning opportunities, we summarize results that point to significant progress in the students' algebraic thinking, in relation to generalization processes that were fostered at different points of the interaction. Qualitative comparative methods were applied to develop themes that are central to what takes place in whole group, marking the positive effect of interaction on the mathematical activity. We discuss data coming from two episodes to illustrate two major themes.

Keywords: learning opportunities, whole group, social interaction, discourse, algebra.

INTRODUCTION

Whole group discussion has become a common practice with a modest amount of research compared to pair and small group work. This report describes a study linking task-related interaction to mathematical learning in whole group. The motivation for the study is the increasing use of collaborative classroom settings, along with the need to refine scientific arguments around them. Following analyses of students interacting in small groups more than twenty years ago (e.g. Cobb, Yackel and Wood, 1991), we aim to examine the potential of large groups from the perspective of creating learning opportunities in the mathematics classroom. For the last two decades, research has pointed to the positive impact of social interaction on mathematical learning (e.g. Brandt, 2007). Other outcomes, however, have pointed to unintended consequences of being involved in interactive settings (e.g. Gresalfi, Barnes & Cross, 2012). We undertake to develop an analysis of large group processes that helps to understand the complexity of learning mathematics in the interaction with *many others*.

In what follows, we present our view of mathematical learning as a socially mediated process, and introduce the methods applied in the research. In the last part of the report we comment on two episodes to illustrate major themes that arise from the analysis of data. A close look at the episodes and the themes leads to a final reflection on the role of social interaction as a mediator in the learning of mathematics. We are aware that several episodes and classroom case studies are required to strengthen any explanation for an educational phenomenon. This is why, although we focus on the unique aspects of two episodes in one classroom, we also attempt to note complexities presumably arising from scenarios with similar didactical and pedagogical orientations.

THEORETICAL FRAMEWORK

In this section we briefly present what we mean by social interaction and mathematical learning opportunities. Then we refer to the domain that frames the objects of teaching and learning in the design research, namely the transition from arithmetic to algebra.

The joint construction of social interaction and learning opportunities

Under the lens of interactionism (Cobb, Stephan, McClain, & Gravemeijer, 2001), social interaction may be defined as oriented human actions that can be developed for performance improvement in situations of teaching and learning mathematics. The ideal of positive interaction as a cultural practice has survived to become a tool for potential learning and for group development. From a vygotskian perspective, it makes sense to interpret the notions of learning and learning opportunities as conceptually and empirically similar. Both refer to the favourable negotiation of circumstances toward the construction of knowledge. Cobb, Yackel and Wood (1991) already suggested the value of thinking of learning at an operational level in terms of what we refer to as Learning Opportunity Environments –LOEs. We view students' learning as a product of their involvement in LOEs, and of their ability to optimize the interaction with others to improve mathematical communication. In our work we focus on LOEs that may contribute to learning situations during the course of students' interactions despite minimum intervention from the teacher. Missed learning opportunities may occur due to the lack of active guidance provided by the teacher, but students have the potential to make the most of many other opportunities by themselves.

The notion of mediation provided by Cole (1998) in the nineties, suggests the mutual influence of different realities in the accomplishment of specific goals and tasks. We see mediation as a culturally-based practice embedded in learning and group development, with the underlying assumption that individuals and groups are ready to (consciously) move their roles and positions in the interaction with others. In our work, mediation is explored in terms of the actions of participants in a mathematics classroom in which the creation of LOEs through peer interaction is expected to lead to "the development of mathematical discussions" (McCrone, 2005). Moreover, our idea of mediation coincides with the selection of certain participants and issues that will receive more attention than others in the interaction. It is this discursive nature that raises a problem for the practice and theory of social interaction as a mediator of mathematical learning: what to do with and how to explain LOEs in which communication is not always successful, and/or is not well addressed in the direction of sharing knowledge. In response to this problem, it makes sense to develop studies that are built on the search for collective situations in which students make the most of a LOE to achieve learning, together with others in which they miss the opportunity at a given point of the interaction. In this report, we focus on the first type of situations.

The difficulties of students in the transition from arithmetic to algebra

As expressed by Kaput (2008), algebraic thinking is the activity of doing, thinking and talking about mathematics from a generalized and relational perspective. In the transition from arithmetic to early algebra, one of the initial difficulties of students has to do with the learning of the language conventions underlying this mathematical domain. Algebraic symbols like an *n* become rather sophisticated, and in the resolution of a problem tend to be misinterpreted and misused. On the other hand, Lins and Kaput (2004) argue that an early favourable start to the learning of algebra is possible by leading the students to foster a particular kind of generality through the use of a problem-solving approach with generalized arithmetic. The reasoning required to solve certain problems can be expanded from concrete arithmetic situations to more complex situations that include the ability to use abstraction. We state that a problem involving several stages of algebraic thinking, from near to far generalization, helps students to adjust their reasoning from meaningful numerical cases to algebraic symbolism and mathematical abstraction. Thus three levels of reasoning may be recommended to represent a problem: a) concrete (considering small quantities), b) semi-concrete (considering big quantities), and c) abstract (using symbols).

Together with the three levels above, the exploration of visual growing patterns is another recommended way to introduce algebraic expressions (Warren, 2000). Many students experience difficulties with the understanding of geometric patterns as algebraic functions. Some of these difficulties come from the lack of appropriate language to describe relationships between variables, the inability to visualize and complete patterns, and the complexity to connect verbal, visual and algebraic representations. Consequently, tasks that encourage visual strategies and relate number and geometric contexts are crucial in the early learning of algebra. For the design of the tasks in our research, we have considered the combination of concrete, semi-concrete and abstract levels of reasoning, and the combination of verbal, visual, numerical and algebraic representations to mathematically model regular situations of change.

RESEARCH CONTEXT AND METHODS

The investigation consisted of preparing and analyzing six lessons in a classroom with a group of students aged 15 to 16, and the teacher. The students were used to pair work and whole class discussion. They were also used to problem-solving dynamics, to listening to each other, and to communicating their mathematical ideas. The research question was: *How does whole group discussion contribute to the creation of mathematical learning opportunities in problem solving classroom environments?*

T-shirt Problem (lesson 4). Since 2009, a brand new T-shirt can be bought, with an added square in the drawing each year. Take a look at the T-shirts of 2009, 2010, 2011 and 2012 above, and find:

- How many white squares and grey squares does the 2015 T-shirt have?
- How can you express the quantities of white and grey squares for any T-shirt?
- How many white triangles and grey triangles does any T-shirt have?

Figure 1. Example of task from the problem sequence

To prepare the design experiment, we elaborated a coherent and focused sequence of six word problems about generalization (see one example in Figure 1) that helped to create a LOE. Coherence was based on the control of a progressive difficulty in the problems from the perspective of algebraic contents, and also on how a problem was mathematically related to the next one in the sequence. For each fifty-minute lesson, one problem was presented by the teacher and then discussed by the students in pairs. The last thirty minutes were devoted to large group work. The teacher acted as a facilitator of the students' interactions, and circulated around the room during pair work. Data collection consisted of audio and video recordings of class discussions.

We began by transforming audio and video files into transcripts. It took time to decide which type of transcription would better suit the aims of the research, while remaining an adequate representation of data with a double emphasis on the interaction and the mathematics. After having examined various options, we looked for key episodes in the videos and elaborated transcripts that illustrated interactional and mathematical features. To determine where transcripts of episodes begin and end, we gave priority to the mathematics. We identified whole group moments in which mathematical practices were at the core of the discussion due to the existence of diverse meanings or the difficulty in understanding a mathematical reasoning. Thematic boundaries and learning opportunities may be differently perceived by different researchers, but the two authors' agreement was guaranteed, along with a third researcher who intervened when it was difficult to reach agreement in the analysis of a specific episode.

Having constructed the set of episodes and reviewed the videos several times, we began a process of comparative and inductive analysis (Glaser, 1969) among episodes from the same lesson and then from the total of lessons. We aimed to elaborate mathematical memos and interactional codes to mark changes in the students' meanings, as well as changes in the direction of interactions. The direction of social interaction depends on whether participants direct their actions toward someone in particular, and whether such actions involve intentions concerning the interpretations under discussion. In an episode with practices of cross multiplication, for example, interactional codes may consider verbal actions by one student aimed at seeking others who share similar ways of making sense of cross multiplication. Other codes may be related to verbal actions aimed at helping each other to understand cross multiplication. There may be codes that point to students asking for clarification of ideas, and so on.

Any social interaction is a combination of interactional codes, and thus it is not possible to have key episodes that are univocally related to single codes. However, an exhaustive attribution of codes to episodes was not attempted. We gave priority to detecting one (possibly two) code(s) that influenced the evolution of the mathematical activity. Next we summarize the direction for two interactional codes that were constructed in advanced phases of the analysis in relation to key episodes. The following codes were developed in an on-going way as the analysis of new episodes contributed different codes and until there was a stable set:

- *Sharing responsibility* -A student follows up a mathematical explanation given by the peer in pair work and gives further information.
- *Expressing confusion* -A student reacts to a prior intervention by claiming lack of understanding with respect to a mathematical reasoning.

Throughout the analysis, interactional codes were completed with mathematical actions. Our use of the term mathematical action echoes the notion of mathematical practice by Godino, Batanero and Font (2007, p. 3): “Any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems.” The attention to mathematical actions led to the elaboration of what we call mathematical memos. There was no limit on the length of memos. Nevertheless, we intended to summarize mathematical characteristics in the actions of students in an episode in about one paragraph (see next section). By providing codes and memos, we advanced toward the construction of themes in order to obtain cases of episodes. The creation of a theme involved specifying two components: i) the mathematical actions at that point in the lesson, and ii) the interactions that appeared connected to the realization of such actions. Emerging themes were expected to inform about learning opportunities in terms of relevant mathematical actions and interactions. Various themes were constructed and, as more data were analyzed, some of them came to assume a greater importance and constituted major themes. Some early themes were either discarded as the analysis progressed, or absorbed into more accurate descriptors. A final group of major themes comprises the findings of the study. In the next section we summarize two of these major themes.

SHARING RESPONSIBILTY AND MAKING SENSE OF ALGEBRA

The excerpt below reproduces part of an episode in which two students are discussing the third question of the T-shirt Problem, “How many white triangles and grey triangles does any T-shirt have?” The T-shirt design pattern refers to an arrangement of squares inside each other. The midpoints of each side of the outer square are joined to make a smaller square inside it and so on. Jose and Gabriel worked together during pair time, and they are now explaining part of the solution in the large group.

Jose: If *n* is the number of the T-shirt, *n* equals the quantity of squares in that T-shirt. The first T-shirt has one square, the second has two, the third has three… But as said before, one square has no triangles. Then you need to take the number of T-shirts minus one, and multiply it by four because each square leads to four triangles. That's the total. If *n* is even, you divide the total by two and get the white and the grey triangles. When *n* is even, you get the same quantity of white and of grey squares. You've taken one out, and get the same quantity of white and of grey triangles.

Gabriel: No, this happens when *n* is odd, no... Yes, when *n* minus one is odd.

Jose: Yes, it is *n* minus one, odd.

Gabriel: When *n* minus one is odd. We made a mistake, when *n* is odd you get one more white square. It leads to the same quantity of white and grey triangles.

Jose: Yes. If *n* is odd, you divide it by two and get the number of the two types of triangles.

While in the first lessons the wording of the problems stresses one of the variables with the algebraic use of the letter *n*, the fourth problem requires the symbolic representation of the variables by the students. Jose uses the *n* to distinguish the location of the T-shirt, established at the start of his reasoning. To reach the unknown quantity of colored triangles, he gives the expression $4(n-1)/2$ for the special particular case of odd *n*. Despite the confusion between even and odd T-shirts, he progressively constructs the pattern with the support of algebraic language and visual thinking. Gabriel points to the issue of even and odd numbers having an influence on the adequacy of the pattern. The two students, however, seem not to be clear about the need to use *n* or $n-1$. In a former episode from the lesson, some students discuss whether the 2015 year T-shirt is the seventh in the collection, while some others consider it to be the sixth due to incorrectly interpreting the year subtraction, 2015 minus 2009. Such a difference also has to do with taking either 1 or 0 as the initial value for *n*. Gabriel looks at the drawing of the third T-shirt and concludes that the pattern by Jose works for the odd cases. He justifies his reasoning on the basis of the characteristics embedded in the general case given by the set of odd T-shirts. In the end, both students manage to make sense of algebra by using the symbolic convention for the generic representation of natural numbers.

The code *Sharing responsibility* represents situations in which a student follows up a mathematical explanation given by the peer in pair work and provides further information. The student feels that s/he should respond to what the peer says and does in some appropriate way. Here Gabriel and Jose share the responsibility of making sense of the algebraic pattern in the context of the T-shirt Problem. This code calls for responsibility based on joint efforts during pair work. For the different lessons, we often see students in the large group behaving as individuals still belonging to a pair structure. Students take more responsibility for what their peers in pair work say in the large group, compared to what other students say and do, for whom the main responsibility is expected to be assumed by the teacher. Although our analysis focuses on whole group and all students showed different ways of participation, the influence of the pair work dynamics appears to be relevant in that it is a locus of responsibility. It can be argued that this sort of student-student collaboration would not be so present if whole group discussion had not been preceded by pair work. In any case, the result in this section calls for the importance of analyzing episodes rather than isolated actions. Individual contributions make sense within the collective situation. However, it seems clear that some interactions in whole group facilitate more learning opportunities than others, as not all of them are aimed at fostering collaboration. The description of memos and codes is not sufficient to understand why this happens. A profound understanding of the theme would require the examination of a broader context including the students' experiences about what it means to participate in large group.

This episode illustrates how group discussion can enhance learning opportunities by making public an error in relation to the algebraic use of n. The intervention by Gabriel makes the incorrect use of the variable explicit, which comes with a clarification on the connections among the value for n, the position of the T-shirt in the sequence and the appropriate pattern. It is our argument that by having the two students sharing responsibility for this discussion in the whole group, the creation of a learning opportunity is facilitated. If Gabriel had not paid attention to his peer, the mathematical error might have been overlooked. In other episodes, *Sharing responsibility* also acts as a mediator in the understanding and manipulation of algebraic expressions. In the same lesson and with respect to the same pair, when discussing the answer to the first question in the problem, Gabriel clarifies some of the words said by Jose concerning the connection between the value for n and the position of the T-shirt. The large group becomes the scenario for this pair to make a contribution to the topic under discussion.

EXPRESSING CONFUSION AND LINKING REPRESENTATIONS

The episode partially reproduced below is an immediate continuation of the previous episode. Both take place in the fourth lesson around the solution of the third question of the T-shirt Problem. To us, the following conversation helps to illustrate the fact that an expression of misunderstanding can become a resource to be exploited by students to optimize the creation of learning opportunities.

Teacher:	Have you understood? Maria [a student], can you explain it?
Maria:	Woops! I have understood nothing!
Jose:	If they tell you that there are three T-shirts, one, two and three… This one, the third, has three squares because each year you have one more square.
Maria:	It will have as many white triangles as grey triangles…

Jose: We'll talk about that later. Now, you have three squares in the T-shirt but one square does not generate triangles, the one in the middle. So you take one out.

Maria: Why do you take one out?

Jose: The one in the middle does not generate triangles. You take these two squares that do generate triangles. You multiply them by four. Each square generates four triangles [pointing to the third T-shirt]

Maria: Okay.

Jose: That's the total amount of triangles. Three squares minus one are two, multiplied by four is eight, the total. You don't know how many are grey and how many white. You only know that eight is the total. But the *n* is odd, and it's the same quantity for white and grey. You divide it by two and get the quantity of white and grey triangles.

A few minutes before, Jose had explained the pattern for the special particular case representing the odd T-shirts, $4(n-1)/2$. He was aware of the two special cases introduced by the particularity of a number being either even or odd. There was an initial algebraic approach to the explanation of the general pattern, that may be viewed as a mere symbol manipulation instead of an expression that shows relationships between the location of the T-shirt in the collection and the quantity of squares and triangles in the design. Such manipulation seemed to hinder Maria's understanding. Jose reacted by linking algebraic language with natural language and including references to the visual context of the problem. On a second attempt to clarify his explanation, he took the case of the third T-shirt, as a generic example to illustrate regularities that support his pattern. The whole episode suggests that Maria came to see the generality in the pattern through the particular case provided by Jose. Jose followed the opposite direction to that developed by him in the construction of the pattern during pair work, in which he had explored particular cases to conjecture generality.

The code *Expressing confusion* represents the actions of a student who reacts to an intervention by claiming lack of understanding with respect to a mathematical reasoning that has been exposed. In this episode, Maria is the student who expresses such confusion. Although the term confusion rather suggests the individual cognitive dimension, it may arise from non-internal reasons such as people poorly communicating their ideas. A student may suppress information that the addressees need in order to make sense of what is being said. As part of an interaction, confusion is to be seen in terms of a collective challenge with people involved in the completion of a shared task. In the context of the theme here, *Expressing confusion* refers to actions that become mathematically productive. A student expressing confusion, however, is not a guarantee of participants exploring the problem-solving processes. Such actions become unproblematic as long as participants are willing to help each other to understand mathematical contents by checking meanings. Understanding is facilitated when communication is seen as possible because participants are considered to be competent. Jose might not have provided examples for his mathematical thinking if he had interpreted Maria's intervention only as an expression of difficulties in the understanding. And vice versa: Maria might not have shown confusion if she had not seen Jose as sufficiently competent to follow her arguments.

We see here group discussion as a LOE in which learning opportunities are provided through making public the incompleteness of a mathematical reasoning. The fact that Maria publicly shares her confusion leads Jose to explain again his reasoning including, on this occasion, newer connections between visual reasoning and verbal patterns. Thus the opportunity to link different representations (verbal, visual, numerical, and algebraic) appears. The actions by the teacher are also relevant in the creation of this LOE. The teacher promotes the tacit demand from Maria, which becomes effective in the configuration of the turns that makes the mathematics evolve. In other episodes from this lesson, the study of the T-shirt sequence reinforces the opportunity to learn that there is more than one correct way to express the same relationship between two variables. This is a common learning opportunity arising from the six lessons: students realizing that two or more differently looking arithmetic expressions are equivalent and mathematically model the same realistic situation.

METHODOLOGICAL ISSUES AND FUTURE PROSPECTIVE

At this point of the report, there are a few methodological issues that need further justification. The identification of mathematical learning opportunities in classroom episodes (which is in turn related to the identification of LOEs) is one of these issues. Our use of the notions of LOE and learning opportunity has been theoretically grounded on the recognition of opportunities as being influential in the promotion of effective mathematical learning. In particular, we have assumed that the awareness of learning opportunities in the classroom discourse is a condition to use them for learning. Thus, throughout the whole process of analysis, the practical problem of identifying evidences of the students' learning has turned into the practical problem of identifying evidences of learning opportunities. It has been our position that the students' learning becomes more or less fostered depending on the participation in LOEs and the exposition to learning opportunities. Consequently, learning can be understood as an increase in the exposition to such opportunities.

But what are the criteria for us to claim the existence of certain learning opportunities? Even though we have not always clear evidence of the students taking advantage of particular opportunities, and experiencing processes of learning something new (Jose, Gabriel, Maria... for instance, may be already aware of a mathematical knowledge and they may be merely reminded of it by another participant), we sustain the idea of identifying potential opportunities for learning as a scientific goal that makes sense in itself. This option brings up the substantial problem of reaching a multiplicity of learning opportunities, some of which do not necessarily contribute to the construction of mathematical learning. We see the current final set of multiple mathematical learning opportunities as the starting point for the development of a second part of the research. Drawing on the same collection of data, we are planning to explore situated connections between learning and learning opportunities. Instead of focusing on concrete 'isolated' episodes, we are considering the longitudinal analysis of sequential aspects of pair work and whole class discussion to trace evidences of learning.

So far, our analysis of classroom episodes has consisted of two main dimensions that have been articulated through the identification of interactional codes and learning opportunities. It is interesting to include a third dimension that helps examine the ways in which the students are supported in their mathematical learning by means of the exposition to certain social dynamics and environments. Such future prospective might serve as a basis to more precisely conceptualize a LOE for the learning and teaching of mathematics (what characterization of learning opportunity environments contributes to better understand the teaching of mathematics?), as well as to construct a typology of mathematical learning opportunities (what differentiation of mathematical learning opportunities contributes to better understand the learning of mathematics?).

Notes

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