CHOICE OF NOTATION IN THE PROCESS OF ABSTRACTION

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This paper focuses on the notation used by students in a study concerning the process of abstraction in constructing a group on the set of symmetries of a square. Participants were asked to identify the elements of this group and to further investigate its various properties by combining two elements at a time and reflecting on the outcome. Particular attention will here be paid to the differences in approach by academic track and vocational track high-school students in embracing the task, a difference which appears to stem largely from the creation and subsequent use of notation for representing the members of the group.

INTRODUCTION

The underlying methodology of the project has been grounded theory (Glaser & Strauss, 1967) which emphasizes the generation of theory from immersion in collected data. This process has so far resulted in 22 clinical interviews with high-school students (aged 16-19) in Oslo, with roughly half of the participants from each of the academic and vocational high-school tracks. Both paired and individual work has been considered, with three of the interviews for each track involving two students working together. *Task based clinical interviews* have been used for data collection, which involve intensive interactions with the individual participant, extended dialogues between interviewer and interviewee and careful observations of the interviewee's work with 'concrete' intellectual objects (Ginsburg, 1997).

As a whole, the study from which this paper derives can be described as an exploration of post-compulsory students' strategies for investigating and organizing abstract mathematical structures when engaging with an unseen task. More specifically it is a consideration of Norwegian students' encounters with the symmetry group of a square. As such there are a number of issues and questions that have emerged, however for the purposes of this paper I will concentrate on the differences in invention and subsequent use of notation by the two groups of students. Some of these differences will then be discussed in light of APOS theory - as a framework for abstraction, and later Duval's analysis of problems of comprehension arising from semiotic representations. The former thus provides an outline of the stages involved in moving from the concrete to the abstract, whereas the latter can be said to explore potential barriers in the transition between these stages.

THEORETICAL BACKGROUND

Piaget proposed the term 'reflective abstraction' to describe the construction of logico-mathematical structures by an individual during the course of cognitive development (Dubinsky, 1991). Reflective abstraction is characterised by the fact that

> It does not proceed from a series of observations of contingent events or objects. Rather, it is a process of interiorizing our physical operations on objects. As we move sets of objects about [...] we interiorize properties of mathematical operations rather than objects; we acquire implicit understanding of commutativity, associativity, and reversibility (Noddings, 1990, p .9).

This idea has featured prominently in Freudenthal's (1973) analysis of mathematical development, where he notes that "the activity of the lower level [...] becomes an object of analysis on the higher level" (p.125). Other mathematics educators working in the Piagetian tradition have called this process 'entification' (Kaput, 1982), 'integration' (Steffe, Glasersfeld, Richards & Cobb, 1983) or 'encapsulation' (Dubinsky, 1991). There are subtle variations within these definitions, but in essence they all describe "the process of forming a static, conceptual entity from a dynamic process" (Gray & Tall, 1994, p. 118). There appears to be widespread recognition, both within cognitive psychology and mathematics education, of reflective abstraction as an essential focus for educators, a key to fostering understanding and development of intellectual thought (Simon, Tzur, Heinz, & Kinzel, 2004), and a crucial skill for a meaningful engagement with mathematics (Hazzan, 2005).

Much of the research on abstraction in mathematics has been carried out with a focus on young children, and the majority of Piaget's own work (on which much of the later research is based) was concentrated on the development of mathematical knowledge at the early ages (e.g. Piaget, Wedgewood, & Blanchet, 1976; Piaget & Pomerans, 1978), rarely going beyond adolescence or basic mathematical structures. He did however suggest that the same ideas could be extended and would still apply for older subjects dealing with increasingly advanced mathematical concepts. This is the context most relevant to this study, and the chosen framework for analysis has therefore been 'APOS theory' which has been developed by a team of mathematicians and mathematics educators led by Ed Dubinsky (see Asiala et al., 1996), and specifically addresses the less investigated avenue of abstraction in more advanced mathematical situations. APOS theory is based on Piaget's principle that an individual learns by applying certain mental mechanisms to build specific mental structures and subsequently uses these structures to deal with mathematical problem situations. APOS is an acronym for Actions, Processes, Objects and Schemas (Dubinsky & Moses, 2003, p. 403), and these can be described as follows:

> 1) A mathematical concept begins to be formed as one applies a transformation on objects to obtain other objects. A transformation is first conceived as an **action**, in that it requires specific instruction as well as the ability to perform each step of the transformation explicitly.
>
> 2) As an individual repeats and reflects on an action, it may be interiorized into a mental process. A **process** is a mental structure that performs the same operation as the action being interiorized, but wholly in the mind of the individual, thus enabling her or him to imagine performing the transformation without having to execute each step explicitly.

Given a process structure, one can reverse it to obtain a new process or even coordinate two or more processes to form a new process via composition.

3) If one becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct such transformations, then we say the individual has encapsulated the process into a cognitive **object**.

4) A mathematical topic often involves many actions, processes, and objects that need to be organized and linked into a coherent framework, which is called a **schema**.

As has been pointed out (Dubinsky & Mcdonald, 2002), these stages need not be linear or even exact descriptions of a learning process. However the theory as a whole provides a useful framework for analysis of qualitative data, and it is in this capacity that it has been and will continue to be used for the purposes of this study.

THE STUDY

Mathematical framework

The concepts that have served as a mathematical framework rest largely on group theory and will be introduced here in brief. Formally a mathematical group can be defined in the following way (Smith, 1998):

A set G equipped with a binary operation $\cdot$ is called a group if all of the following axioms are satisfied:

- Closure: For every a, b $\in$G their product a $\cdot$ b is in G.
- Associativity: For every a, b, c $\in$G it follows that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- Identity: There is e $\in$G, such that $e \cdot a = a \cdot e = a$ whenever a $\in$G.
- Inverses: Given any a $\in$G, there is b $\in$G such that $a \cdot b = e = b \cdot a$.

The group that has played a particularly central role in this study is the symmetry group of a square, which is a member of the 'dihedral groups' (the groups of symmetries of a regular polygon), often denoted D4. This group has been suitable for interview purposes for a number of reasons: It is unfamiliar to the students, and yet it involves a range of concepts that undoubtedly (implicitly or explicitly) form part of school mathematics, such as rotations, reflections, symmetry, identity and inverses, just to mention a few. Furthermore, the group has eight elements - a number large enough that it is not necessarily immediately obvious what these are, and yet small enough that within a reasonable time students can find these elements themselves by inspection, rather than being presented with them; the task is simultaneously attainable and challenging. Moreover, the group as a whole is not commutative. That is, a $\cdot$ b (a followed by b) is not *necessarily* the same as b $\cdot$ a (b followed by a) for two elements a and b in the group.

It should be emphasized that although group theory has been utilized as an underlying framework for the project, it has never been made explicit to the students. Rather it has provided points of reference along the way in the design of tasks and in the analysis of the data, much as APOS theory has been, and will continue to be, used as a supportive framework in the process of analysis.

The interview process

Interviews (lasting up to 2 hours including short breaks) were recorded on a laptop web-camera. The reason for this choice was that a laptop is such a familiar part of everyday life that it is far less obtrusive than a video camera being pointed at the participants. Moreover it is easy to carry around between interviews, and has left no improvement in quality of image or sound to be desired.

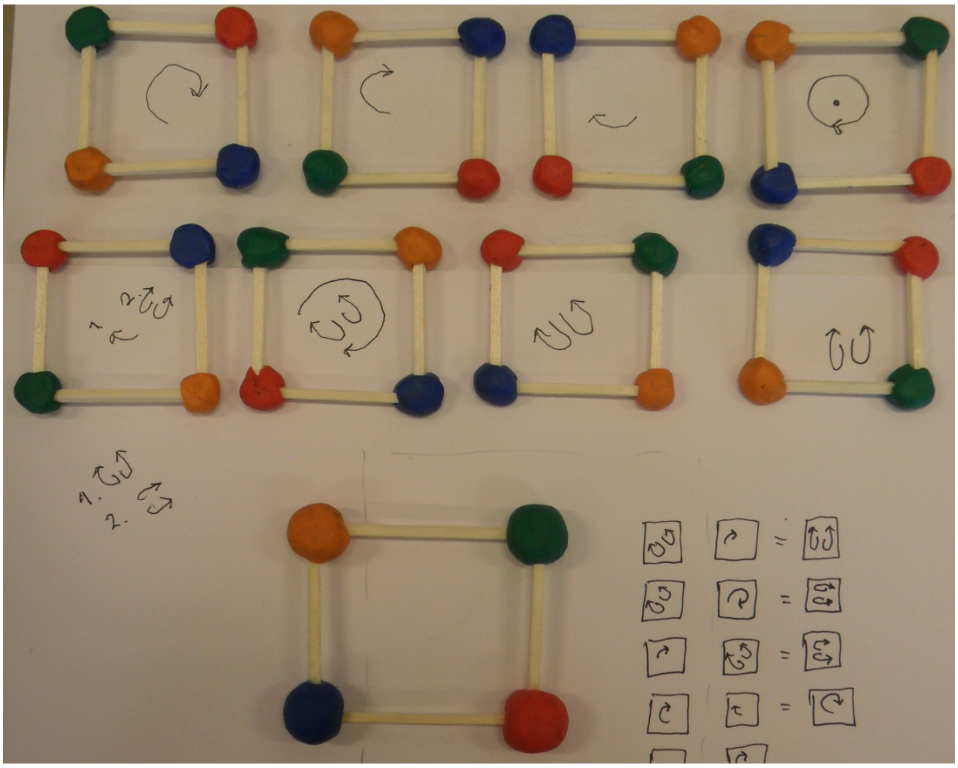

Figure 1

The interviewees had pen and paper available and were encouraged to think out loud and to write down their ideas. They were presented with a square with coloured corners (see figure 1), and were asked to 'play' with this figure under certain 'rules' which commanded firstly that it would have to end in the plane, and secondly that the symmetry of the figure had to be preserved. Each time a new move was performed, the students would record it by placing a smaller version of the square on the paper, showing the new position of the colours, and they would further describe, using notation of their choice, the motion that had been performed The square was always returned to its original position before the next move, and participants were asked to find as many configurations as possible. When they were happy with the number of possibilities found, the students were asked to combine pairs of moves, writing down what they did as they went along, and to comment on the results. The relationships between the elements of this new 'system' were discussed, during which the students were encouraged to lead the conversation, and they were subsequently given the freedom to explore any aspect of the system that caught their attention. For the initial record kept by one of the interviewees, see figure 1.

PRELIMINARY RESULTS

Students' initial engagements with the square were mostly at a concrete level, with attention paid to its physical appearance and the position of the colours. However, while identifying all the elements of the group was largely an action based endeavour, investigations became increasingly abstract and towards the end of their interviews many students were touching upon concepts such as inverses and identity elements, identifying subgroups and looking for overall behaviours of the system. For example, many participants established that D4 as a whole is not commutative, and additionally tried to determine a pattern for this 'non-commutativity'. Several students considered questions such as: which elements commute? And perhaps more strikingly: When two elements fail to commute, what is the 'difference' between the outcomes? And, is this difference always the same for non-commuting elements? It transpired that this 'difference' is always one of 180 degrees, a result with which I was unfamiliar before the interviews. Indeed, while this and other student discoveries merit further elaboration, limitations of space prevent it. I mention them to illustrate clear shifts from physical actions and attributes towards abstract structures.

However before reaching this level of abstraction, participants had to choose some notation for the elements of D4 that they identified, so that this notation could later be used when operating with the members of the group (i.e. when combining two elements as shown in figure 1). It is this specific aspect of the investigations that will be presented here, and it was largely in this process that a marked difference could be perceived between the academic and vocational tracks.

Figures 2a and 2b show the notation chosen by an individually interviewed academic and vocational track students respectively.

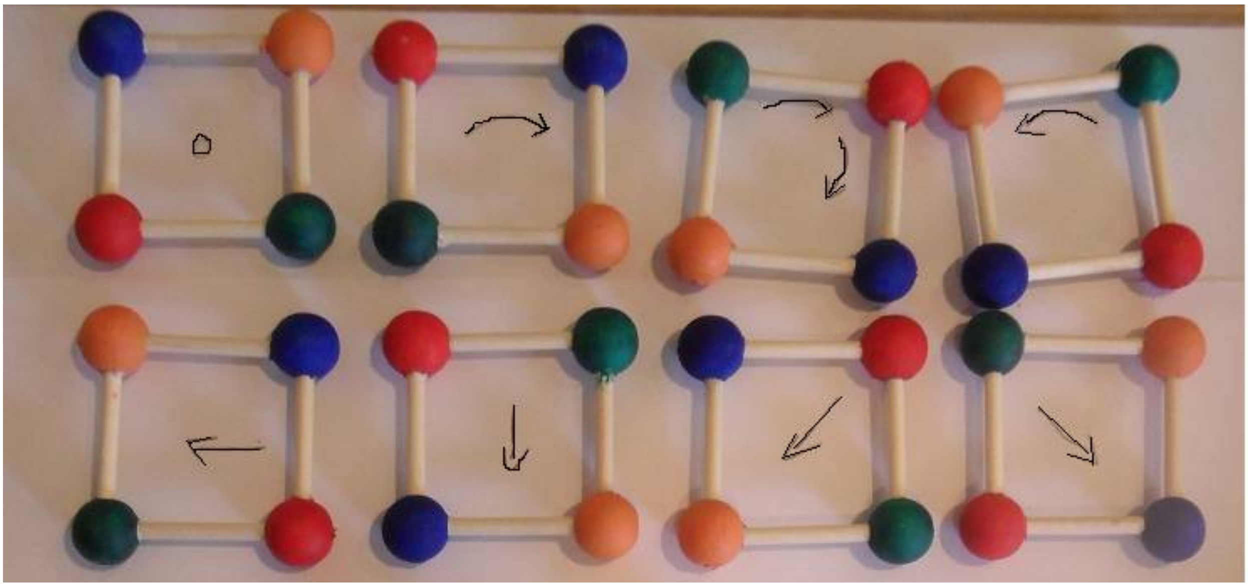

Figure 2a: Academic track student (individual interview

The contrast is rather striking and serves to highlight an issue that is recurrent in the interviews. One might argue that the use of notation is not central in itself; however the problem seemed to be that when the students started combining elements, a good notational structure facilitated explorations of the behaviour of the system under consideration, which in turn appeared to ease the transition from the concrete to the abstract mode of thought. However the student with the notation shown in figure 2b was simply stuck. He could not remember what his symbols meant, and the wordy descriptions were too cumbersome to use.

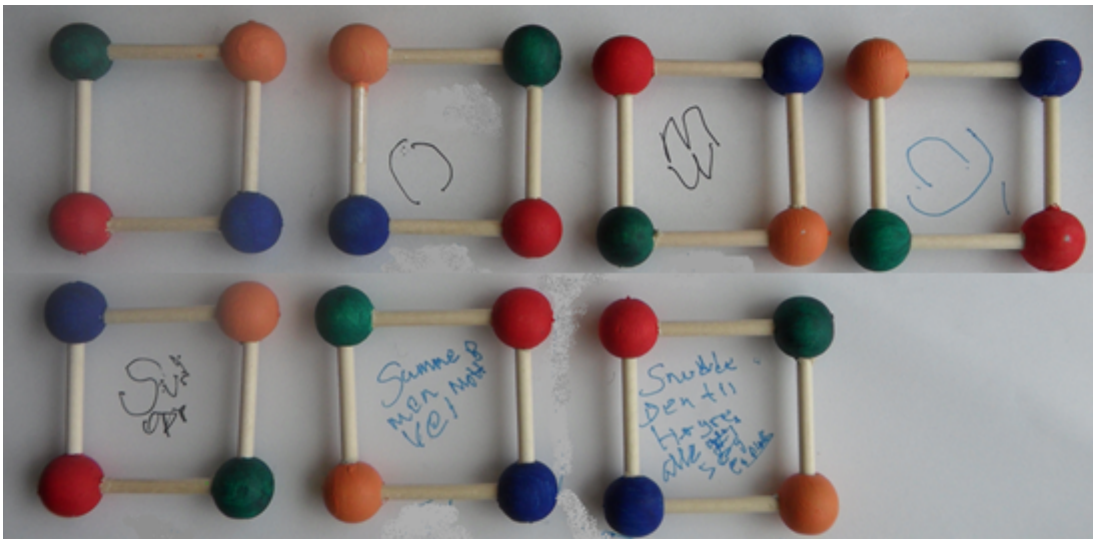

Figure 2b: Vocational track student (individual interview)

Similar differences were found in the paired interviews; the academic track notation was essentially identical to that in figure 2a, and an example of paired vocational track notation can be seen in figure 3. What the photos do not show, however, is the length of time taken by vocational track students inventing their notation. In this context the paired interviews provided some useful insights - while the individual interviews were conducted largely in silence, the paired interviews involved negotiation (which was substantial for the vocational track) accessible to me as interviewer. The following extract from a vocational track interview shows two students trying to agree on a way to describe a diagonal reflection.

Figure 3: Vocational track students (paired interview)

Alex: For this one green and blue swap places.

Mona: hmm... what if you picked a description that was independent of the colours?

Jonathan: Nah... that will be hard...

Alex: flip it down to the right

Jonathan: maybe use the edges then

Alex: yeh, if we don't use the colours and corners we have to use edges ... ok ... so if a person wanted to describe it to you ... what would be the easiest way so you'd get it [asking Jonathan]? For me it would be to take the top right bit and pull it down to the left bottom bit...

Jonathan: what right and left though? ... I like using a, b, c, d and draw a picture of a square and show what goes where.

Alex: yeh... but what if you don't have a picture of the square...

After further elaborate negotiations and various drawings, their final result was that drawn in the third square from the left on the second row in figure 3 (The writing translates to: 'swap places with each other').

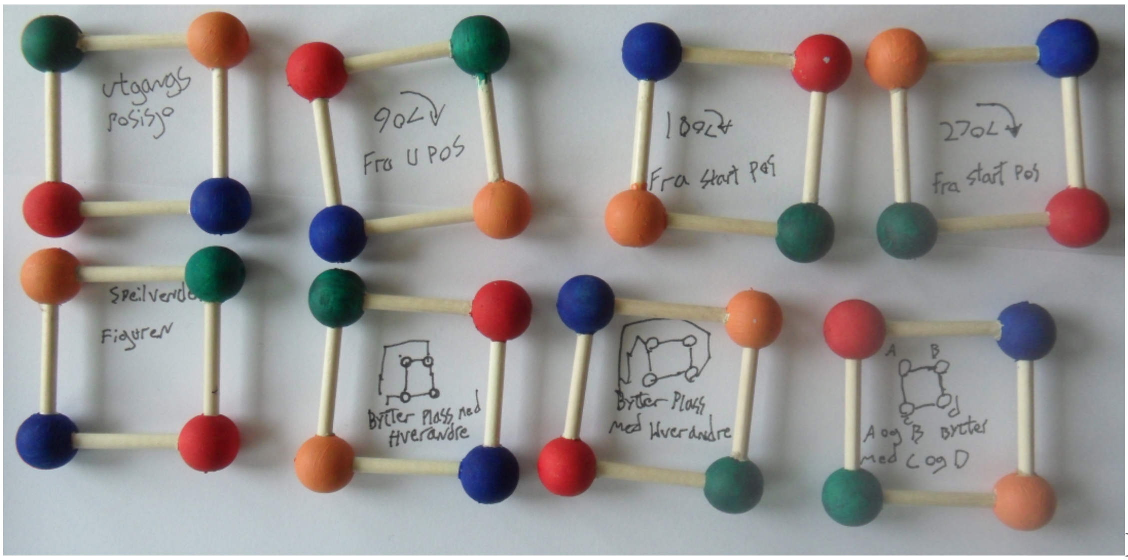

Figure 3

DISCUSSION

So what exactly are the differences between the chosen notations? Coherence and simplicity appear to be central issues. The academic track students tended to choose the same 'kind' of notation for the elements of D4 (such as seen in figure 2a). This gave them an overview of the number of elements; for instance, once they had drawn one diagonal arrow, they naturally realised that the other 'diagonal move' would also give a member of the group. Moreover these simple arrows, where they continued to use them, facilitated further exploration of the system, particularly when pairs of moves were combined. The vocational track notation tended to lack both coherence and simplicity. As shown in figure 2b, there was no 'single kind' of notation, but a mixture of drawings and words, which often hindered the identification of all the elements. Moreover, as shown in figure 3, many students insisted on keeping a small drawing of a square as part of each notation. This underpinned their lingering dependence on the physical attributes of the concrete artefact, and additionally made the subsequent exploration of the system slow and cumbersome to keep track of.

It should however be noted that problems of notation were not limited to the vocational track students. The academic track participants were all fairly confident and coherent in the *invention* of notation; however it is important to distinguish between mere invention and invention followed by appropriate application. To illustrate this, the work in figure 4 might be considered.

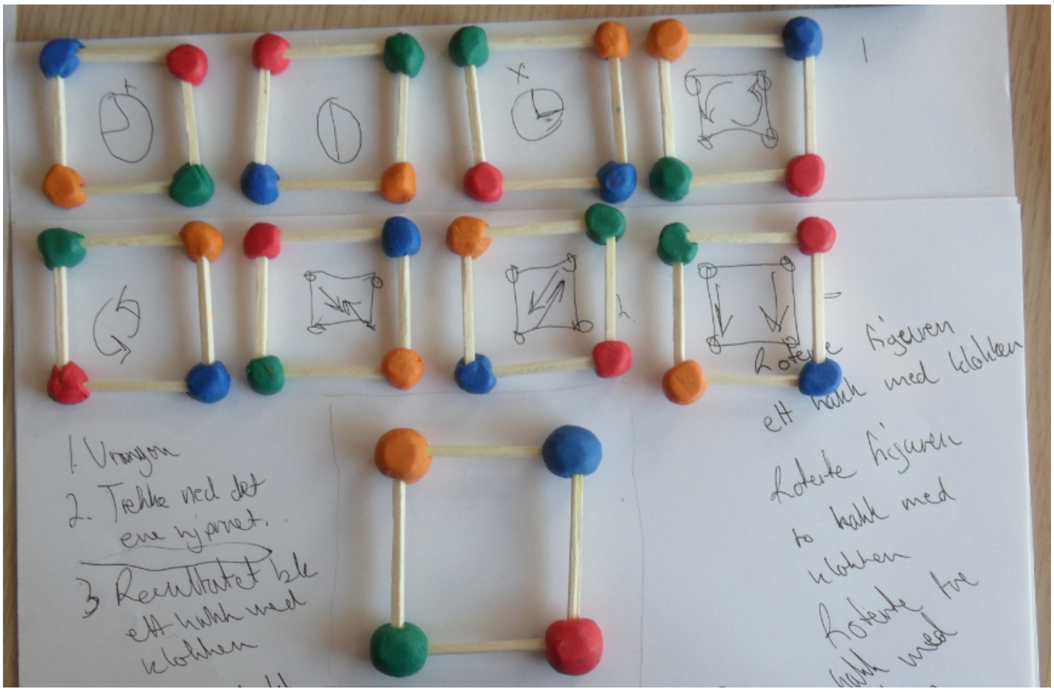

Figure 4

It can be seen that the student has succeeded in creating largely coherent and descriptive symbols for each element, however when he moves on to combining pairs of elements below, he does not think to *use* the symbols, but rather writes his operations and results out in words, a process which is time consuming and inefficient for exploring the abstract structures of the system further.

In the context of APOS theory, the issues described above are closely related to the first two steps of *action* and *process,* and how they can lead to *object* (AP → O). The preliminary observations and analysis of collected data suggest that at these stages serious barriers might occur, and that the speed at which these barriers are overcome depends largely on the invention and use of notation. (Once guided to overcome these difficulties, the explorations by both sets of students were rather impressive). The study thus indicates that it might be appropriate to consider APOS theory in conjunction with theories of semiotics, where for instance Duval contends that

> representations can […] be signs and their complex associations, which are produced according to rules and which allow the description of a system, a process, a set of phenomena. There *the semiotic representations, including language, appear as common tools for producing new knowledge and not only for communicating any particular mental representation*." (Duval, 2006, p. 104, italics added)

Moreover he explicitly asks: "What cognitive systems are required and mobilized to give access to mathematical objects and at the same time make it possible to carry out multiple transformations that constitute mathematical processes?" (p. 104). Here it could be argued that a different way of posing this question might be: what cognitive systems lend themselves to facilitating student's transitions between the stages proposed by APOS theory in the process of abstraction?

One answer might be that signs and semiotic systems of representation are needed, not only to label objects at the action stage, but to promote the transition to thinking in terms of processes and encapsulating these processes into objects in their own right. Now whereas this idea has been researched in various ways in relation to the learning of basic mathematical structures by young children, there appears to be less focus on the interplay between theories of abstraction and theories of semiotics in more advanced mathematical situations.

The study presented here, though very much a 'work in progress', suggests that more careful consideration ought to be given to the semiotic resources drawn upon by older students, and that the ways in which these resources are used (or neglected) can give insight into barriers that students meet in the later stages of abstraction. Although further analyses will be necessary, these preliminary results suggest there is reason to believe that the students who struggled to invent and use notation appropriately during the task under consideration, might also struggle in other mathematical situations, as symbol use and creation is crucial in the development of abstract mathematical proficiency, particularly in algebra where Norwegian students are repeatedly shown to perform poorly (Gronmo, Onstad & Pedersen, 2010).

When considered against the backdrop of other research on students' proficiency (or lack thereof) in abstraction and algebra (e.g. Breiteig & Grevholm, 2006) this study suggests a need to improve the situation. Initiatives and approaches such as suggested in the 'Trip Line' (an algebra module designed for the high-school curriculum as part of part of 'The algebra Project, Inc.', 2008), might well be worthwhile to pursue, as it unlike many current teaching plans pays explicit attention to the invention and use of symbols in algebra and thus integrates Halliday's (1993, p. 93) proposition that a good strategy for understanding human learning in general, would be to pay attention to "how people construe their resources for meaning - how they simultaneously engage in 'learning language' and 'learning through language'".

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