THE EXTENT OF CRITICAL AND CREATIVITY THINKING DISPLAYED DURING PROBLEM SOLVING AMONG STUDENTS ATTENDING THE MATHEMATICALLY TALENTED YOUTH PROGRAM

Einav Aizikovitsh-Udi

Beit Berl Academic College, Israel

This pilot study investigates the extent of mathematical creativity among 57 eight-grade talented students in the Mathematically Talented Youth Program. The reasoning these students is examined when solving a problem, as is the degree of mathematical creativity and aesthetic in their approach in solving a non-routine mathematical problem. The analysis explored whether the students' mathematical thinking was dependent solely upon previous mathematical knowledge and skills. The majority of the students relied on technical algorithm to solve the problem. Although talented students coped well with the thinking challenge, most of them operated at a very basic level of creativity. There is a need to broaden and develop mathematical-logical thinking as an integral part of instructional programs in mathematics.

Keywords: Mathematically talented students; Mathematical reasoning; Mathematical creativity, Mathematical aesthetics, Non-routine problem solving

INTRODUCTION

This pilot study examined the mathematical aesthetic in students' problem solving solutions. The aim of the study was to undertake a preliminary investigation of the extent of mathematical creativity and aesthetics in solving non-routine problems, among students in the Mathematically Talented Youth Program. The study was conducted to answer following research questions:

1. What kinds of reasoning did the talented students use in solving a non-routine problem?

2. What was the degree of creativity and aesthetics in problem solving among students who attended the Mathematically Talented Youth Program?

3. According to the degree of creativity, can we conclude that mathematical thinking among the students of the Mathematically Talented Youth Program is dependent solely upon previous mathematical knowledge and skills?

BACKGROUND

Non-routine Problem-Solving

A problem is defined as a relatively new and complex situation, in which the problem solver must invent a strategy to generate the solution. Here, unlike in the repetitive exercises or algorithm-based problems, there is no given path for the solution (Lester, 1980). A non-routine problem presents the learner with a novel, unknown situation, and stimulates his or her imagination and senses, and offers a challenge – motivating the desire to succeed. Solving non-routine problems should be the heart of teaching and learning mathematics. Solutions to non-routine problems can be reached in various ways. According to Silver (1997) and Ervynck (1991), solving problems in various ways is a tool for both evaluating and developing mathematical creativity and aesthetics.

Mathematical Creativity

Mathematical creativity in school mathematics is usually connected with problem solving or problem posing (e.g., Silver, 1997). Ervynck (1991) posits that mathematical creativity in problem solving is the ability to formulate mathematical objectives and find their innate relationships; it is the capacity to solve problems according to the appropriateness of integrating both the nature of logic-deduction in mathematics education and its evolved concepts into its core. According to Silver (1997) and Ervynck (1991), mathematical creativity has two characteristics, general and specific. General original or creative thinking enables the ability for problem-solving or problem-posing in various fields, creating innovative and original solutions of high quality. These ideas or solutions are usually elegant and surprising. Such thinking is characterized by mental flexibility, curiosity, a well-developed imagination, high interest or motivation in finding solutions, the creation of metaphors, and goal-oriented thinking.

Mathematical Aesthetic

Sinclair (2004) identifies three roles of aesthetic in mathematical inquiry: evaluative role, generative role, and motivational role. Evaluative role concerns the "aesthetic nature of the mathematical entities and is involved in judgments about the beauty, elegance, and significance of entities such as proofs and theorems" (Sinclair, 2004, p. 264). Generative role "involves nonpropositional, modes of reasoning used in the process of inquiry, (which is) ... responsible in generating new ideas and insights that could not be derived by logical steps alone" (Sinclair, 2004, p. 264). Motivational role refers to "the aesthetic responses that attract mathematicians to certain problems and even to certain fields of mathematics" (Sinclair, 2004, p. 264). Sinclair (2004) contends that in the education settings, much of the emphasis of aesthetic has been exclusively on the evaluative role.

Studies on Talented Youth in Mathematics and Science

According to Allan (1991), a basic assumption of a talented student is the realization of potential. The learner's mathematical potential is a complex function of ability, motivation, and learning opportunities. The realization of students' potential and the nurturing of excellence will allow for the training of a new generation of scientists and creative artists who will contribute to the development of society, sciences, and technology in the modern age. Equal opportunity is a basic condition here, not in the sense of identical learning materials, but rather in the sense of identical opportunities for students to realize their personal potential, discover interests, and find personal motivation. The researchers Shore and Kanevsky (1993) quote studies that state that talented or gifted students have distinctive thinking characteristics. Some of these differentiate the thinking of talented students from that of ordinary students: a wide memory and knowledge base, self-control mechanisms (meta-cognition), quick thinking, problem presentation, and mental flexibility.

Studies on non-routine problem solving among talented and/or gifted youth

Various studies (Stepien & Pike, 1997; Boyce et al, 1997) report that problem-based learning is a new approach in the education of talented and gifted students that is becoming popular in the USA. Many studies have demonstrated that the thinking and learning characteristics of gifted students are different from those of other students (Hong & Aqui, 2004). Creative students in mathematics are more cognitively resourceful than their peers who achieved high grades in school mathematics (Hong & Aqui, 2004). In order to fully realize their potential, talented students need opportunities for more rapid and deep learning that focuses on topics connected to their fields of interest.

METHOD

Subjects of this study consisted of 57 eight-grade students attending the Mathematically Talented Youth Program. They were in the second year of the program. The Mathematically Talented Youth Program is open to students who excel in mathematics and have passed the Bar-Ilan University entrance examination. The curriculum was developed by the Israeli Center for the Advancement of Mathematical Sciences at Bar-Ilan University, with the support of the Ministry of Education. The program's aim is to have the students take their matriculation examination in mathematics (5-point matriculation—the highest level) in the tenth grade, allowing them to study university level mathematics in the eleventh and twelfth grades. The goals of the program are:

- To provide enrichment math classes for excellent students across the country in order to encourage them to pursue advanced academic studies in all fields and to progress in these studies.

- To nurture talented youth from across the country in the period prior to their academic studies in mathematics while they are still in high school – motivating them to obtain college degrees in the sciences (including medicine, economics, life sciences, etc.)

The curriculum for the eighth, ninth and tenth grades includes mathematical enrichment topics and topics required for academic studies in mathematics and natural sciences.

The Instrument

A questionnaire with one mathematical task was used in this study (Figure 1). The single item tested depth of thinking based on the strategy in which the problem was solved. This question was taken from a set of enrichment examination published by the Technion Israel Institute of Technology (2005). There were several reasons for choosing the item. First, minimal preliminary mathematical knowledge was required in order to arrive at the solution. Second, a straightforward (unsophisticated) solution exists. Third, the problem was set in a familiar context involving a daily situation (sharing cake in a family).

Mother made a square-shaped cake and decorated it. She cut the cake into four pieces (See below). The youngest daughter got the square with the sun. The oldest son got the piece with the moon. Father got the two pieces with the hearts. Who ate more of the cake – the father or the son? Explain your answer.

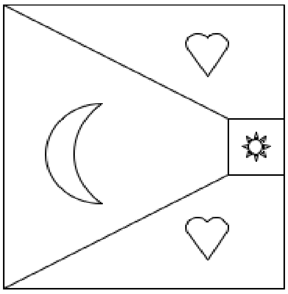

Figure 1. The mathematical task given to the students (Technion, 2005)

POSSIBLE ANSWERS/SOLUTIONS:

Geometrical Method

In Figure 2, the four triangles marked with * (triangles AEK, AGK, DLF, and DLI) are congruent and thus have the same area. Rectangles EBCF and GHJI are congruent because they have the same width (the side of the small square KHJL) and the same length (the side of the big square ABCD). Since both rectangles share the same square KHJI, the remaining pieces of the rectangles should have the same area (i.e., rectangles EBHK and LJCF have the same area as rectangle GKLI). Therefore, the son's piece (trapezoid AKLD) is the same as the sum of the father's two pieces (trapezoids ABHK and DLJC).

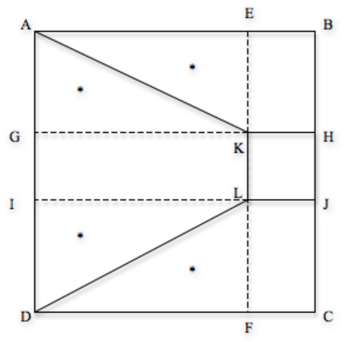

Figure 2. A geometrical solution

Algebraic Method

We mark the length of the cake "a" and the side of the small square "b". Using the area formula of a trapezoid (or a variation of the same formula) $\frac{(b_1+b_2)}{2}\cdot h$, the area of the son's piece is $\frac{(a+b)}{2}\cdot(a-b)$ or $\frac{(a+b)(a-b)}{2}$. The area of one of the father's piece is $\frac{(a+b)}{2}\cdot\frac{(a-b)}{2}$ or $\frac{(a+b)(a-b)}{4}$, so two pieces have the area of $\frac{(a+b)(a-b)}{2}$. Therefore, both the father and son's pieces have the same area.

Visual Manipulation Method

When we move the small square along the right side of the cake, the area of the large trapezoid does not change because the bases and the height are maintained. The sum of the areas of the small trapezoids also does not change either, because the sum of their heights is constant. Thus, the solution does not depend upon the position of the small square. With the small square at the corner, the father's portion and the son's portion are congruent.

PROCEDURE

The questionnaire was handed out to three classes of eight-grade students taught by three different instructors in the Mathematically Talented Youth Program in central Israel. These students were in their second year of the program. There was no time limit for the students to solve the problem. There was no explicit instruction to the students as how they should approach the problem, and they were allowed to use any tools they need, including calculators, to help them solve the problem. They were encouraged to write their explanations on how they came up with the solution and to provide justifications of their thinking. All their written responses were collected for analysis.

Data Analysis

Students' written responses were first analyzed qualitatively to identify the types of reasoning used to solve the problem. The number of correct responses for each type of reasoning was calculated. The responses were also coded for the level of creativity based on Ervynck's (1991) framework: Level 1 (employing algorithms), Level 2 (developing a method from a situation), and Level 3 (constructing a solution by exploring what is stated in the problem). The percentage for each level was also calculated. Finally, only students' solutions that scored high on the level of creativity (Level 3) were analyzed for the level of aesthetics using Dreyfus and Eisenberg's (1986) framework: Level of prerequisite knowledge; Clarity; Simplicity, brevity and conciseness; Structure, power, cleverness, and elements of surprise.

RESULTS

As expected, majority of the students (about 96%) provided correct answer to the problem, and only about 3% answered incorrectly. In the first part of this section we present four types of reasoning that emerged from the students' responses and their distributions. Representative examples were taken from the students' responses. In the second part of the section, we present the levels of creativity of the students, the levels of aesthetics for the selected sample, and their distributions.

Types of Reasoning

After examining the students’ answers, the following four types of reasoning were found:

1. *Explanation based on calculating the area using plane geometry*: The students referred to axioms and definitions of plane geometry, writing the solution as a geometrical proof, with a theorem and an explanation. A correct link was made between the problem and plane geometry.
2. *Explanation based on area calculation*: The students related to the geometrical figures as given shapes, such as rectangles or trapezoids, as part of the problem's data. They used known formulas to calculate the areas.
3. *Explanation based on verbal area description*: Students referred to the problem as a collection of areas of given geometrical figures. Areas were related to as equal and overlapping and were "cut out" and moved around.
4. *Other*: Explanations that do not fit any of the above categories. For example, by employing estimation strategies rather than calculation or argument.

The majority of the students (about 40%) answered the thinking challenge using area calculations where the solutions are based on a completely technical algorithm. 28% of the subjects answered the thinking challenge with a verbal explanation containing explanations of the problem as a collection of given geometrical figures, overlapping areas, equal areas, “cutting out” areas and moving them around. Approximately 16% of the subjects answered the thinking challenge using a geometrical proof where they referred to definitions, axioms, and proofs in plane geometry and wrote the solution as a geometrical proof according to theorem and explanation. A similar percentage was categorized under "Other".

Analysis according to "level of creativity"

The majority of the subjects, (40.4%), were found to be at the most basic level of creativity –using the algorithm-based solution. (Calculating areas using simple formulas). 17.5% of the subjects who solved this thinking challenge were found to be at the basic level of creativity, developing a method from a given situation, relating to overlapping and equal areas, "cutting out" areas and moving them around. 15.8% of the subjects were found to be at the third level of creativity, the highest level. These students knew how to examine the problem in an educated manner, that is, they related to the given figures as not pre-defined, proving them, and using their proofs in their solutions. During the analysis some answers were categorized as "other" because they did not fit to any of the other categories. 26.3% of the subjects were included in this category.

Analysis of the findings in terms of aesthetics

The degree of aesthetic in the students' solutions was evaluated qualitatively using Dreyfus and Eisenberg's (1986) characterization of aesthetic values of a problem solution: (a) reliance on minimum preliminary knowledge; (b) importance of clarity;(c) simplicity, brevity, and conciseness; and (d) cleverness of the solution and element of surprise. Only subjects who scored in the high level of mathematical creativity (N = 9) were included in this analysis.

Reliance on minimum preliminary knowledge.

None of the subjects scored a 2, "Relies on minimum prerequisite knowledge," in this category. The students relied heavily on broad mathematical knowledge such as definitions, theorems, and axioms. Because the context of the problem hinted examination of the areas for comparison ("*… who ate more ...*"), the students relied on their knowledge of finding areas by partitioning into regular shapes they were familiar and in which they know the area formulas (square, rectangle, and trapezoid).

Clarity

A high degree of clarity was found in the subjects' solutions to the thinking challenge. The arguments that students in this group provided tended to be simple and straightforward. The focus was on linking the problem to definitions, theorems, and axioms, and relating to known formulas of area of regular shapes.

Simplicity, Brevity, and Conciseness

A high degree of simplicity was found in all aspects of the solutions, with the exception of questionnaire 24A, where the solution consisted of substantial number of steps and complex argument.

Cleverness of the solution and the element of surprise

Despite the fact that in this study all the subjects were talented students, we did not find the element of surprise, with the exception of one response. This solution was categorized as high in creativity because the student related the height of the trapezoid AEFD and the sum of the heights of the two trapezoids ABHE and DCGF because in both cases they are missing the length of the square. Although this transitive reasoning shows cleverness, it did not fully satisfy the demands of criteria for the element of surprise since there was still a reliance on the formula of trapezoids.

DISCUSSION

Problem solving is a significant means of developing mathematical understanding that includes interest and enjoyment. It is also a vehicle to express mathematical creativity and aesthetics that are often neglected, especially when working with talented students, and even with the general student population. The development of mathematical creativity requires solid foundation of mathematical knowledge (Meissner, 2000) and its transformation into new knowledge (Nakakoji, Yamamoto, & Ohira, 1999) because excellent knowledge in content helps individuals to make connections between different concepts and types of information (Sheffield, 2009). This strong foundation knowledge provides the basis for flexibility to move within concepts and between concepts. Thus, the extent of prior knowledge is prerequisite to how new information will be organized and determines the degree to which such information will be explored (Sheffield, 2009). Therefore, it comes to no surprise that creative thinking is more inherent among students who exhibit mathematical accuracy and fluency, especially in the context of working with non-routine and novel mathematical tasks, requiring them to pose original and meaningful solutions (Binder, 1996). However, other researchers proposed that creative potential contribute to the improvement of mathematical knowledge, suggesting that students who use mathematical content in creative ways lead to further mathematical learning (Starko, 1994). These two views lead to the old age the chicken or the egg causality dilemma. With no intention of resolving this philosophical conundrum, this paper assumes the double-helix nature of creativity and mathematical content knowledge, suggesting a synergistic interaction of the two.

CONCLUSIONS

The level of difficulty of the thinking challenge presented to the students in this study was appropriate as shown by the majority of the students answering the question correctly. However, the type of reasoning, the level of creativity, and the degree of mathematical aesthetics varied greatly among the students.

Three types of reasoning emerged among the talented students in the program: analytical reasoning, practical reasoning, and creative reasoning.

Majority of the students in the program (about 40%) were at the most basic level (Level 1) of mathematical creativity. Although students can be categorized as talented, but the level of mathematical creativity may not necessarily

The students in this study did not seem to develop a high level of mathematical aesthetics when solving the problem. Students relied heavily on prerequisite knowledge and although presented their solutions with high clarity, they failed to express simplicity, structure, and cleverness in their solutions. Furthermore, there lacked elements of surprise in their solutions.

Although the research literature on talented students support the notion that talented students are more creative, no specific level of creativity was found in this segment of the population.

There are several limitations to this pilot study. First, the use of a single item to assess the students' level of mathematical creativity and aesthetic was problematic. Given its function as a pilot study, this investigation presented a glimpse of the students' ability and the results were not meant to be conclusively. A more comprehensive set of questions is deemed necessary to provide a more coherent picture of their capability. It is intended that a more extensive set of mathematical tasks will be developed, incorporating a wider variety of mathematical and figurative contexts and requiring a variety of methods for their solution. These tasks will form the basis of a more extensive study into the development of student mathematical creativity. In addition, cognitive interviews with the students would be beneficial to gauge students' thinking process and examine their decision for choices regarding specific strategies. It is anticipated that such interviews will be an integral component of a more extensive investigation. The interview will inform us whether these students are solving the problems from a practical stance, which is evidence in their lack of attention to the aesthetic value of the solutions.

TEACHING RECOMMENDATIONS AND FURTHER STUDIES

Understanding the development of mathematical creativity and aesthetics among gifted and talented students is crucial to initiate and support their growth. Future studies should examine instructional support and classroom practices that afford the development of mathematical creativity and aesthetics over a period of time during their program. Teachers need to emphasize not only creative ways to solve problems, but also the elegance of the solutions because that is what mathematicians do. This can be done by providing time and attention to the process and structure of mathematics.

REFERENCES

Allan, S. D. (1991). Ability-grouping research reviews: What do you say about grouping and gifted? *Educational Leadership, 48*(6), 60-65.

Binder, C. (1996). Behavioral fluency: Evolution of a new paradigm. The Behavior Analyst, 19, 163-197. Retrieved from http://www.abainternational.org/TBA.asp

Boyce, L. N., VanTassel-Baska, J., Burruss, J. D., Sher, B. T., & Johnson, D. T. (1997). *Journal of the Education of the Gifted, 20*, 363-379.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *For the Learning of mathematics, 6*(1), 2-10.

Ervynck, G. (1991). Mathematical Creativity. In: D. Tall (Ed.), *Advanced Mathematical Thinking* (pp.42-53). Dordrecht: Kluwer Academic.

Hong, E., & Aqui, Y. (2004). Cognitive and motivational characteristics of adolescents gifted in mathematics: comparisons among students with different types of giftedness. *Gifted Child Quarterly, 48*, 191-201. doi:10.1177/001698620404800304

Lester, F. (1980). Research on problem solving. In: R. J. Shumway (Ed.), *Research in mathematics education.* Reston, VA: National Council of Teachers Of Mathematics.

Meissner, H. (2000, August). Creativity in mathematics education. Paper presented at the meeting of the *International Congress on Mathematics Education*, Tokyo, Japan.

Nakakoji, K., Yamamoto, Y., & Ohira, M. (1999). A framework that supports collective creativity in design using visual images. In E. Edmonds & L. Candy (Eds.), Proceedings of the 3rd conference on *Creativity & Cognition*, pp. 166 – 173. New York: ACM Press. Retrieved from http://www.informatik.unitrier.de/~ley/db/conf/candc/candc1999.html

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM, 3*, 75-80.

Sinclair, N. (2004). The role of the aesthetics in mathematical inquiry. *Mathematical thinking and learning, 6*(3), 261-284.

Sheffield, L. (2009). Developing mathematical creativity-questions may be the answer. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in Mathematics and the Education of Gifted Students* (pp. 87-100). Rotterdam: Sense Publishers.

Shore, B.M.,& Kanevsky, L. (1993). Thinking processes: Being and becoming gifted. In K. A. Heller, F.J. Monks, & A. H. Passow (Eds.), *International handbook for research and development on giftedness and talent* (pp. 133-148). London: Pergamon.

Starko, J. A. (1994). Creativity in the classroom. New York: Longman.

Stepien, W. J., and Pike, S. L. (1997). Designing problem-based learning units. *Journal for the Education of the Gifted, 20*, 380–400.

Technion-Telem (2005). *Number 3 Unified Examination for Ninth Grade students Studying Toward a 5-point Matriculation Examination in the Program for Realizing Mathematical Excellence: 19 FILL IN Year Part 3 Enrichment*, Haifa.