TEACHING PRACTICES AND MODELLING CHANGING PHENOMENA

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While much research has demonstrated the positive impact of mathematical modelling on student learning, considerably less research has focused on the teaching practices that are needed to support modelling approaches to student learning. In this study, we examined the characteristics of teaching in a classroom setting where the students engaged in a sequence of model development tasks designed to support their abilities to create and interpret models of changing physical phenomena. The results illustrate the demands that modelling tasks place on teachers, ways of responding to those demands, and suggest needed pedagogical shifts in teaching practices.

Over the past three decades, much research has focused on the potential of mathematical modelling to impact student learning throughout K-16 mathematics. But despite the evidence from research on the positive impact on student learning, progress has been slow in the widespread adoption of mathematical modelling as a classroom practice (Blum & Borromeo Ferri, 2009; Maass, 2011). While many factors influence changes in schooling, one crucial factor in any kind of change in classroom practices is the teacher (Godwin & Sutherland, 2004; Ruthven, Deaney & Hennessy, 2009). But the teaching practices associated with mathematical modelling have received somewhat limited attention from researchers (c.f., Blum & Borromeo Ferri, 2009; Doerr, 2006; Lingefjard & Meier, 2010; Maass, 2011; Wake, 2011; and others). As we have argued elsewhere (Doerr, 2007; Doerr & Lesh, 2011), the knowledge needed for teaching mathematics through modelling and applications appears to differ in some significant ways from traditional approaches to teaching mathematics. In this paper, we examine the nature of teaching practices that support student learning through mathematical modelling, and thereby contribute to an understanding of the knowledge needed for teaching mathematics through modelling. To that end, we investigated the characteristics of teaching in a pre-college classroom setting where the students engaged in a sequence of model development tasks to support their abilities to create and interpret models of changing physical phenomena.

THEORETICAL BACKGROUND

Over the last twenty years, researchers have documented the difficulties that students encounter in learning to create and interpret models of changing phenomena (Carlson et al., 2002; Michelsen, 2006; Thompson, 1994). To address these difficulties, we designed a model development sequence (Doerr & English, 2003; Lesh et al., 2003) to support the development of students' abilities to model changing physical phenomena. This modelling approach to student learning is what Kaiser and Sriraman (2006) identify as a "contextual modelling" perspective, emphasizing tasks that motivate students to develop the mathematics needed to make sense of meaningful situations. Many researchers have used model eliciting activities (MEAs) developed by Lesh and colleagues (Lesh et al., 2000; Lesh & Zawojewski, 2007) to investigate the development of students' conceptual systems (or models) in a wide range of settings and contexts. In this study, however, we moved beyond a single model eliciting activity to design a model development sequence whose goal was to produce a conceptual system (or model) that can be used to make sense of a collection of structurally similar physical world contexts.

A model development sequence begins with a model eliciting activity, in this case, designed to confront the student with the need for the construct of average rate of change. This construct is central to students' abilities to create and interpret models of changing phenomena. The MEA is followed by one or more model exploration activities and model application activities (c.f., Doerr & English, 2003; Lesh et al., 2003). Model exploration activities focus on the underlying structure of the elicited model and on the strengths of various representations and ways of using representations productively. Model application activities engage students in applying their model to new situations, which can result in further adaptations to their model, extending or deepening understandings of representations, and refining language for interpreting and describing the context. In this study, the model development sequence was intended to support the development of the students' generalized understanding of average rate of change and their abilities to create and interpret models of changing phenomena. Throughout model development sequences, students are engaged in multiple cycles of descriptions, interpretations, conjectures and explanations that are iteratively refined while interacting with other students and participating in teacher-led class discussions.

The diversity and complexity of the multiple cycles of the development of the students' models places substantial knowledge demands on the teacher, as teaching "becomes more open and less predictable" (Blum & Borromeo Ferri, 2009, p. 47). As Maass (2011) found in her study, responding to the openness of modelling tasks was especially challenging for teachers with a "static" disposition with its strong focus on final examinations and on teacher-centered pedagogies. The openness of modelling tasks, which is in part a consequence of the diversity of student thinking, leaves the teacher with the need to develop strategies to support the students in making progress with the task, but without directly showing the students how to resolve their difficulties (Lingefjard & Meier, 2010). However, as Lingefjard and Meier note, "it is obviously not enough to ask the teacher to avoid giving a solution to their problem" (p. 106). What the teacher needs is a range of strategies to draw on and, just as importantly, a set of rationales that will enable her to interpret the events of the classroom, select tasks to further the development of students' models, and engage students in the self-evaluation of their models (Doerr, 2007). Such a range of strategies and rationales would provide the basis for responding to students without doing the task for them, or as Blum and Borromeo Ferri (2009) characterize it, for maintaining the balance between providing sufficient guidance for the students while preserving student independence. Characterizing such strategies and elaborating their underlying rationales by the teacher is the focus of this study. To that end, our study was guided by the following question: what are the characteristics of the teaching practices that support students' abilities to create and interpret models of changing physical phenomena when engaged in a model application activity?

METHODOLOGY

This study used design-based research as an approach to studying teaching and learning in the classroom with the intent of contributing to theories of teaching while producing outcomes that are useful in naturalistic settings (Cobb et al., 2003). This design experiment began with the collaborative framing by the researchers and the teacher (third author) of a model development sequence that was intended to support students in developing their concept of average rate of change and in creating and interpreting models of changing physical phenomena. We first describe the model application activity that occurred near the end of the model development sequence, elaborating the intended learning goals for the students. We then describe the context in which the teaching occurred and the iterative cycles of analysis that occurred to understand the teacher's actions in the classroom and her interpretations of those actions.

In the model application activity examined in this paper, students were asked (1) to create a model of the intensity of light with respect to the distance from the light source, (2) to analyze the average rates of change of the intensity at varying distances from the light source and (3) to describe the change in the average rates of change as the distance from the light source increased. The students were given flashlights, meter sticks and light sensors to use with their graphing calculators to measure and collect data of how the light intensity varied with the distance from the light source. Light intensity changes with the distance from the light source at a non-constant rate and can be modelled by an inverse square function. In earlier model exploration tasks, the students had explored the patterns of change in linear and exponential functions. However, the patterns of change for an inverse square function are not approximated by either of these patterns, and hence students needed to draw on other known functions for this context and provide a rationale for choosing a function. Since all of the students had taken a prior course in physics, it is nearly certain that they had studied the inverse square law that applies in this situation. However, we did not assume that the students had investigated or understood or would recall that the reason for the inverse square law in this context is related to the geometry of the sphere. Thus, having students make sense of the relationship to the surface area of the sphere and ways of representing that relationship was part of this model application activity. In addition, students needed to apply their model of average rate of change in a context where the independent variable was not time, but was distance. Finally, students had to interpret negative average rates of change in terms of the light intensity and distance. Overall, this model application task required the students to apply and extend their concept of average rate of change to a new context of changing physical phenomena.

CONTEXT AND PARTICIPANTS

The sequence of model development tasks formed the basis for a six-week course for students who were preparing to enter their university studies. The teacher had four years of experience teaching secondary and college students; this was her third year teaching the summer course. There were 35 students in two sections of the course, all of whom had volunteered to participate in the study. Eleven of the students were female and 24 were male. All students had completed four years of study of high school mathematics; 21 students had studied calculus in high school and 14 had not studied any calculus. Pairs of students completed the model application activity over the course of three lessons. Throughout their work on the task, the teacher led several whole-class discussions that involved students in discussing their emerging models of light intensity and representations of how that intensity changes. The teacher also engaged in conversations with pairs of students as they created their model of light intensity and considered how the intensity changes with respect to distance.

DATA SOURCES AND ANALYSIS

The data sources included videotapes of all class sessions, written field notes and memos, class materials such as worksheets and a record of board work, the teacher's lesson plans and annotations made by the teacher during the lesson. Following each lesson, there was an audio taped debriefing session with the teacher, which captured the teacher's reflections on the lesson and any changes to the plans for subsequent lessons. The model application activity took place over three lessons; each lesson lasted one hour and 50 minutes. Our analytic approach was a collaborative examination of the teacher's actions in and interpretations of classroom events. The analysis of the data took place in two phases. Consistent with the iterative approach of design-based research, the first phase of analysis took place during the six weeks of teaching. In this phase, the research team met with the teacher and regularly engaged in discussion about the model development sequence, the progress of the class as a whole, and our observations about students' thinking about average rate of change and their use of mathematical representations for expressing their ideas. Analytic memos were written by members of the research team to document their emerging understandings of the teaching practices and observations about student learning.

In the second phase of the analysis, members of the research team viewed the videotapes and wrote a detailed script of each lesson, identifying the nature of the teacher's activity and the teaching dilemmas that occurred in each lesson. Following the principles of grounded theory (Strauss & Corbin, 1998), codes were developed to categorize the teaching practices. As we analyzed the practices, we sought confirming and disconfirming evidence in the teacher's lesson plans and annotations during the lesson, and with the teacher's perspective on the lesson from the de-briefing interviews. We present two of the results of our analysis of the teaching practices that supported the students as they created their models: (1) revealing and revising student ideas; and (2) developing and refining representations. We also discuss the difficulties encountered by the teacher in balancing the tension between guiding students and maintaining their independence.

RESULTS

Revealing and revising student ideas

Throughout the model application activity, the teacher engaged the students in revealing and revising their ideas about how the light intensity changed as the distance from the light source increased. This episode occurred at the beginning of the activity. The teacher asked the students about their intuitive ideas on the changes in light intensity, based on their everyday experiences with light. She posed the following question: "Imagine the tail lights of a car moving at a constant speed away from you. Is the light intensity (1) fading at a constant rate, (2) fading slowly at first and then quickly, (3) fading quickly at first and then slowly, and (4) unsure." The students responded to this question using a student response system (also known as voting systems or "clickers"), with the results shown in Table 1. The teacher routinely used the option of "unsure" to discourage students from guessing and to encourage students who see difficulties or ambiguities in a question to continue thinking, without being forced to choose a particular response.

Responses	*Number and Percent Response*
Fading at a constant rate	8 57%
Fading slowly then quickly	4 29%
Fading quickly then slowly	1 7%
Unsure	1 7%

Table 1: Student responses to the rate at which light intensity changes

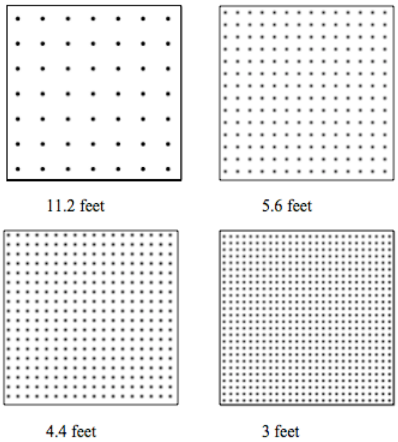
The teacher engaged the students in a discussion about their reasoning and found that students had several different perspectives on this context. One student argued: "no matter how big the light is, you can see it at different distances"; this argument suggests that the light intensity does not change with respect to distance. Several students offered an argument in support of fading at a constant rate by reasoning that "the speed of the car is constant." Others focused on the constant speed of light and reasoned that even though the constant speed of light is different from the constant speed of the car "it's like running in a train" where one can simply add the speeds. However, some students were sceptical about the relationship of the constancy of the speed of the car and of the speed of light to the intensity of the light. One student posed the question: "the car moves constantly, but how do you see the light?" and another asked: "we're talking about intensity. How does that relate to the speed of light?" This discussion was entirely an argument among the students, and revealed their reasoning about how the intensity of light changes at different distances from the light source.

At this juncture, the teacher stepped in, leaving their arguments unresolved, and gave them a task that would enable them to evaluate and potentially revise their ideas. The teacher signalled this as she said: "We are going to sort this out." She gave them data collection equipment that they could use to measure light intensity at different distances from a point source of light. The teacher deliberately did not discuss their ideas further because, as she later commented, she did "not want to give it all away." Rather, she intended for the students to engage in collecting and analyzing data that would enable them to answer this question for themselves. By collecting and graphing data, the students evaluated the alternatives and came to the resolution that the intensity of light decreased at a non-constant rate as the distance from the light source increased.

Developing and refining representations

In keeping with the methodology of design-based research, this model application activity was developed through several iterations. In previous versions, we found that students encountered difficulties in developing meaningful representations of the change in light intensity with respect to distance from the light source. Hence, as we began this modelling task with the students, we explicitly focused on students' images of light intensity. In the lesson following the data collection, the teacher posed a question, analogous to that in the previous episode, but intended to further develop students' representations of light intensity. The students were asked to interpret a "dot" representation of intensity at various distances from a flash bulb and to find the intensity at two unknown distances (see left side in Figure 1). The students initially had difficulty understanding and using this representation. The teacher then introduced the table representation shown on the right in Figure 1. The students recognized that an equation fitting this data would be useful, as one student commented that we "need an equation, but we don't know what it would be."

At this juncture, the teacher polled the students to find out which parent graph they thought would best correspond to the table of data, thus revealing (as in the previous episode) students' ideas about a possible symbolic representation. Most of the students focused on two of the answers, with 57% (n=8) choosing an exponential function, 29% (n=4) choosing $y = 1/\sqrt{x}$, and with 7% (n=1) each choosing y=1/x and $y=1/x^2$. As before, the teacher asked the students to resolve the question of finding an appropriate equation to fit the data. Using their graphing calculators and working with partners, the students rejected $y = 1/\sqrt{x}$ as a parent graph. Two pairs of students came up with two distinct functions: $y = 1400(1/x^2)$ and $y = 715(0.58)^x + 12$, both of which fit the given data reasonably well. However, this response from the students had not been anticipated by the teacher in her planning and left her uncertain, in the moment of teaching, as to how to proceed.



distance	*intensity*
3	169
4.4	81
5.6	49
11.2	12.25

Figure 1: A dot and table representation of light intensity

Unlike the previous episode, where the teacher knew that collecting and graphing data would enable the students to evaluate and revise their ideas about the changing intensity of light, it was less clear how to engage the students in a critique of these two functions, especially since both functions were a reasonably good fit of the data. The teacher juxtaposed the projection of the graph of each function and the data, and turned the question over to the students, asking "which [function] makes more sense?" Several students saw the exponential function as "more accurate" and one student argued that the graph of $y = 1400(1/x^2)$ would show up in the second quadrant and hence "wouldn't be accurate to the data." Still uncertain as to how to engage the students in a critique of these functions, the teacher re-polled the students as to which parent function would best model the data. This time, 86% (n=12) of the students chose an exponential function and 14% (n=2) chose $y=1/x^2$. Re-polling the students gave the teacher some additional time to think about how to proceed; during this time she quickly conferred with a member of the research team who suggested focusing the students' attention on the long term behaviour of both functions. The teacher linked the long term behaviour of the function to the students' intuitions that the intensity of light should get "closer and closer to zero as we get out further and further." This led them to reject the exponential decay function, which did not approach zero.

For the teacher, this somewhat partial resolution was critically important, since the inverse squared behaviour needed to be understood as a meaningful and explanatory representation of the change in light intensity with respect to distance, not simply as a "good fit." However, the issue we wish to raise is that knowing how to further the students' own thinking, in the moment of teaching, was neither obvious nor easy from the perspective of the teacher. In this episode, as the teacher ended the lesson, she focused the students' attention on the critical question of *why* an inverse squared representation was reasonable. She said that the "thing I want you to think about is 'why'? Why does this inverse square function make sense in this situation?" To answer this question, the students would need to further develop their ideas about representing how light intensity changes in terms of the distance from the light source.

In the next lesson, the teacher again focused the students' attention on making sense of how light intensity is changing with respect to distance. She began by asking the students about "why it [an inverse square function] would make sense?" and "How do you think about light coming out of a light source?" Several students responded with ideas about light going in "all directions equally," "travels evenly," and "in all directions." The teacher pursued these ideas and asked: "what image do you think of when you think of all directions equally?" One student offered an image of rays: "near the point source, they are really close. But then they go apart. … As they [the rays] get farther from the point source, they get farther from each other. … And that's why the intensity is less." Several other students offered an image of "spheres" moving out from the light source.

The discussion continued as the teacher built on these images, with student generated representations of enlarging spheres and re-visiting the dot-based representation of intensity; this discussion eventually led to the formula for the surface area of the sphere. The students had moved from the dots representation, to a table representation (both shown in Figure 1), to a symbolic representation, to images of rays and spheres, and to the formula for the surface area of a sphere. At this juncture, the teacher was again faced with deciding what to do next. Rather than guide the students through bringing these ideas together, the teacher turned these elements of representing their model of changing light intensity back to students, asking them to think "about all these ideas and put some of this together … One of the questions is why do you think light behaves this way [as an inverse square]?" She encouraged them to use the representations that had been discussed as "ways to reason about that" and thus develop and refine their representations of changing light intensity.

DISCUSSION AND CONCLUSIONS

This study began with the design of a model development sequence intended to develop students' abilities to create and interpret models of changing phenomena. The design of that sequence provided multiple opportunities for students to generate and revise ideas, to interpret and give meaning to various representations, and to reason about changing phenomena. When students are engaged in such modelling activities, teachers are likely to encounter substantial diversity in student thinking, including student approaches that can not be fully anticipated. This places substantial new demands on teachers to respond with strategies that support students in making progress with the modelling task, but without overly directing students in resolving their difficulties. The results of this study highlight two teaching practices in response to those demands. First, explicitly revealing the diversity of student intuitions about changing phenomena provided an opportunity for the teacher to engage students in revising their ideas. Second, supporting students in developing and refining their representations of changing phenomena enabled the teacher to make visible representations that could be useful in providing explanatory descriptions about the behaviour of changing phenomena. In engaging in these practices, the teacher encountered moments that caused tension in knowing how to productively proceed in the lesson without simply directing the students toward some known solution. In both cases, the teacher made an important pedagogical shift from a practice of teacher evaluation of student thinking to engaging students in the self-evaluation of their ideas. Such a practice is well aligned with preserving student independence (Blum & Borromeo Ferri, 2009).

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