FERMI PROBLEMS INVOLVING BIG NUMBERS: ADAPTING A MODEL TO DIFFERENT SITUATIONS

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1 ABSTRACT

In this paper we describe a classroom experience based on the sequencing of Fermi problems related to estimating large quantities. The models used by a group of Compulsory Secondary Education (16 year-old) students are described herein. The variations the students apply to the models in order to adapt them to similar problems formulated in different contexts are shown. In the conclusions section, we reflect on this didactic proposal and the possibilities it offers to the students, so that they assimilate and internalize the models worked on.

2 INTRODUCTION

In this study we introduce Fermi problems oriented towards the estimation of large quantities. We understand that the modelling processes which appear in the solution of this type of problems with a realistic context cannot replace real-life decision-making, however their use in the classroom promotes attitudes which can be useful for everyday life (Jurdak, 2006). The problems presented in our study are based on the estimation of large numbers. We herein describe the way students adapt their modelling strategies to different problems with similar mathematical structures but which are formulated in different contexts. Seldom found as class problems, solving them demands the students to create their own strategies, adapt them and use them to solve different problems. This leads them to identify with and include these strategies in their mathematical knowledge base, which according to Schoenfeld (1992) may help students become competent problem solvers.

3 THEORETICAL REFERENCES

3.1 Modelling

One of the most relevant scientific activities involves creating models which provide an abstract recreation of objects, phenomena or processes we wish to understand. In recent years, there has been a strong tendency to attempt to approach model creation to the classrooms.

Lesh & Harel (2003) define model as follows:

> Models are conceptual systems that generally tend to be expressed using a variety of interacting representational media, which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs, or experience-based metaphors. Their purposes are to construct, describe or explain other system(s).
>
> Models include both: (a) a conceptual system for describing or explaining the relevant mathematical objects, relations, actions, patterns, and regularities that are attributed to the

> problem-solving situation; and (b) accompanying procedures for generating useful constructions, manipulations, or predictions for achieving clearly recognized goals. (p. 159)

According to this definition, a model can be understood as an abstract way of representing a particular phenomenon or reality. The way students elaborate models in order to solve problems is a matter of discussion and different views exist on this subject (Borromeo Ferri, 2006). However, in general terms, it is agreed to be a multi-cyclic process. According to Blum (2003), modelling processes can be structured into five main stages: i) Simplifying the real problem into a real model; ii) Mathematizing the real model into a mathematical model; iii) Searching for a solution from the mathematical model; iv) Interpreting the solution of the mathematical model and v) Validating the solution within the context of the real-life problem.

3.2 Estimation and Fermi Problems

When we intend to answer questions such as: how long would I take to get to the train station? How many 5kg paint cans do I need to paint the walls of my flat? Or, how many teaspoons of sugar do I need to cover the 250 grams indicated in the recipe? We need to make estimations. By estimation we mean a rough calculation or judgement of the value, number, quantity or extent of something. After revising the literature, three types of estimation can be found: numerosity, estimation of measurements and computational estimation (Hogan & Brezinski, 2003). This fact is due to the existence of a wide range of tasks which, although they don't share the numerical patterns which enable their execution, all require the concept of estimation (Booth & Siegler, 2006). Numerosity refers to the ability to visually estimate the number of objects arranged on a plane; measurement estimation is based on the perceptive ability to estimate length, surface area, time, weight or similar measurements of ordinary objects, while computational estimation refers to the process by which the value of a calculation, such as 13.2÷4.3+6.91, is approximated.

There are yet two types of activities that are referred to as estimation for which we haven't found any relevant studies in the field of Mathematical Education. On the one hand, in a branch of Statistical Inference, estimation is known as the group of techniques which allow calculating an approximate value for a population parameter from the data provided by a sample. On the other hand, another mathematical activity regarded as estimation is the calculation of values obtained either from predictive activities or from approximating a reality by using a model which represents a situation. A good example of this kind of situations is directly reflected in what are known as Fermi problems.

Ärlebäck (2009) offers the following definition of a Fermi problem:

> Open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations. (page. 331)

Ärlebäck (2011) states that working with Fermi problems can be useful to introduce modelling in the classrooms for several reasons. Indeed, we have confirmed that they don't require any specific type of previous mathematical knowledge, the students are obliged to estimate several quantities by themselves (since the problems don't provide numerical data) and are encouraged to discuss the issue with their peers.

4 THE GOALS OF THE STUDY

Based on a class activity carried out with students in their 4th year of Compulsory Secondary Education, using a sequence of Fermi problems aimed at the estimation of large quantities, we study the models the students create to solve the problems as well as how they adapt the models to other similar problems. Thus, the goals of the study are:

1. To identify the models students use to solve problems

2. To identify the modifications they introduce to previously applied models when facing new problems

5 METHODOLOGY

The work dealt with in this paper is part of a wider study presented in Albarracín & Gorgorió (2011) where we research individual strategies for solving Fermi problems aimed at the estimation of large quantities. These types of problems are not currently included in the Spanish curricula and are not usually worked on in class, which means that students aren't being taught specific methods by which to solve them. Thus, students are obliged to create their own resolution strategies which include models of the situations proposed. By definition, the resolution process of Fermi problems can be based on breaking them into smaller problems, which should be easier for the students to approach than the original problem. Dealing with large quantities doesn't allow for simplistic approaches to the solution, and thus we expect the students to come up with strategies richer in mathematical elements, from the abstract representation of the studied reality.

The experience presented hereafter was carried out on a group of 22 pupils in their 4th year of ESO (compulsory secondary education, 16 year-olds) with no previous instruction in modelling. This time we asked the students to solve a set of Fermi problems in teams, which required estimating the number of objects distributed over a surface area in contexts initially familiar to them. The first problem refers to the school itself, while the following four problems require information which isn't directly available to the students, and which they would have to obtain from an external source. The problems are the following:

- Problem A: How many people fit in the school playground?
- Problem B1: How many people fit in a concert at the Palau St. Jordi[1]?

[1] Palau St. Jordi is a pavilion built for the Barcelona'92 Olympic Games

• Problem B2: How many people fit in a demonstration held in the town hall square of your city?

• Problem B3: How many people fit in a demonstration held in Plaça Catalunya (Barcelona)?

• Problem B4: How many trees are there in Central Park?

The problems were worked on in several sessions. In the first session, the students were set problem A and asked to write an individual resolution proposal. Afterwards, the students were arranged into work teams of 3 or 4 (6 groups in total) which had to come to an agreement on a group resolution proposal and determine the actions and resources required to estimate the number of people that fit in the school playground. In the second session the students executed the previously planned work and started to write their reports. In the third session, the reports were completed and they shared the results and methods used.

In the fourth session, the students were presented with the following four problems (B1, B2, B3 and B4) and were allowed to access the internet if necessary. The purpose of the fifth session was to complete the different resolutions and produce a second results report. The sixth session was carried out as a conclusive activity in which the results were compared with information obtained from external sources.

The data used in this study are the reports created by the students and the observations collected by the first author during the experience. We herein present some of the data collected. Fig. 1 shows the resolution of some students who propose estimating the amount of people that would fit in the high school playground by counting the number of them that would fit in it if arranged into rows and columns.

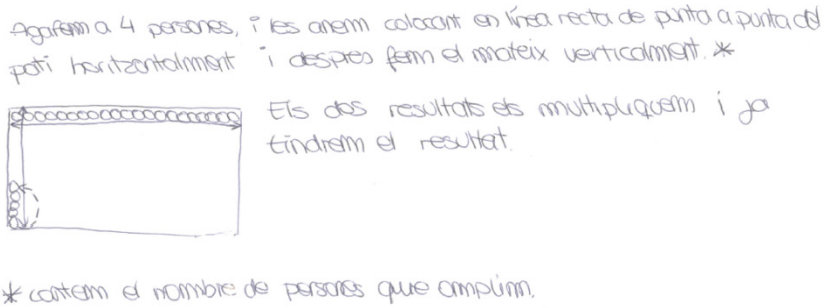


Fig. 1. An example of the resolution of problem A.

The abovementioned proposal reads “ We would take 4 people and place them in a straight line from one end to the other of the playground crosswise and lengthwise, counting the number of people that we fit into it. We would then multiply both results.” As displayed in Fig. 2, the students have made their calculations and obtained 26 and 82 people respectively for each dimension of the courtyard, for which they give a result of 2,132 people.

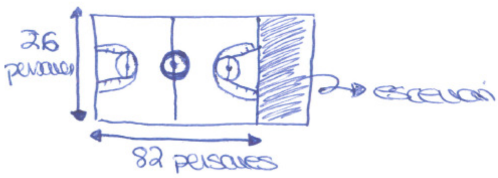


Fig. 2. Data collected for the resolution to problem A.

In Fig. 3 we present some students' resolution of problem B2, which requires estimating the maximum amount of people that would fit in the town hall square during a demonstration.

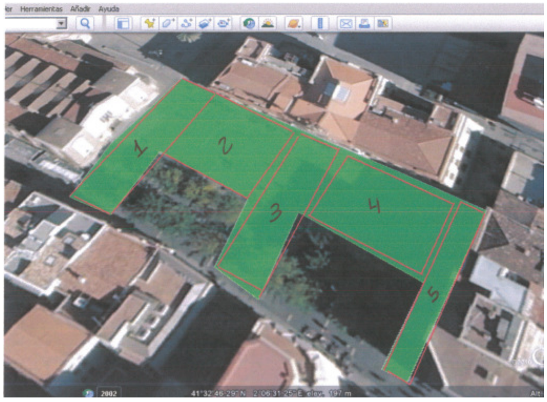
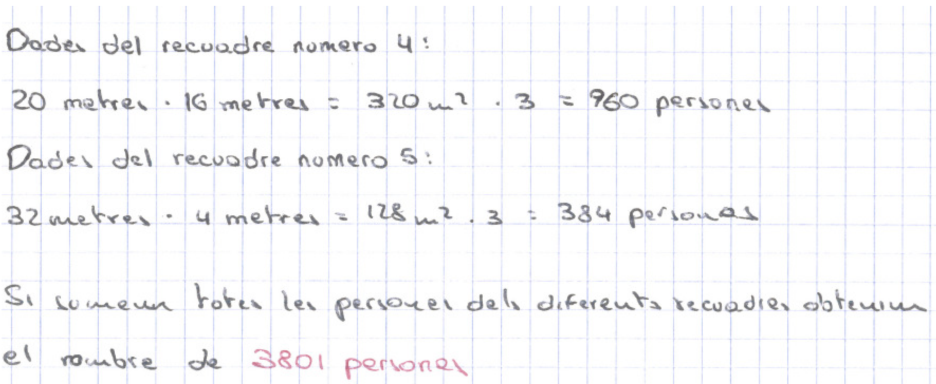


Fig. 3. Available space in the town hall square.

The caption of this picture is the following: "We searched for the town hall square on Google Earth and marked the areas where people could be in with the polygon drawing tool. We assumed they wouldn't tread on the green areas. We afterwards divided the shaded area into rectangles to make it easier to find out their surface areas. We assumed 3 people could fit in a square metre." After that, the students calculated the number of people that could stand in each of the separate areas with their previously-obtained surface data. Figure 4 shows the calculations for zones 4 and 5, as well as the final result.


Dades del recuadre numero 4:

20 metres · 16 metres = 320 m² · 3 = 960 persones

Dades del recuadre numero 5:

32 metres · 4 metres = 128 m² · 3 = 384 persones

Si sumem totes les persones dels diferents recuadres obtenim el nombre de 3801 persones

Fig. 4. Calculations done for the resolution of problem B2.

The data was analyzed following the model presented in Albarracín & Gorgorió (2012), which identifies the resolution strategies proposed by ESO students for several Fermi problems which require estimating large quantities. Within the scope of our research, we understand a resolution strategy as a plan of action or policy designed to achieve a major or overall aim.

The analysis is centred on describing the specific actions the students propose and placing them in more general settings. For instance, some students propose counting the number of people attending a demonstration one by one, by asking all of them to write down their name on a list, or suggest recording video footage of a leakage in order to count the number of drops falling throughout the recording by hand. Both of these proposals to diferent problems portray different plans of action but show the same intention, which is to carry out an exhaustive count of the entirety of objects in the problem. Therefore, these proposals show diferent actions with the same kind of plan, which we interpret as adaptations of different problems to the same type of strategy.

By using the quantitative data analysis software NVivo 8, we established different analysis categories corresponding to the strategies detected. These categories are: lack of strategy, exhaustive count, use of an external source of information, reduction of the problem to a smaller one, comparison with a real-life situation and breaking the problem into different parts to be solved separately.

The latter strategy contains elements of modelling. The way the students break up the main problem into different sub-problems is determined by how they represent the situation studied. Several models have been identified for each situation, which portray different ways into which the problem can be broken up. Some of these models coincide with those identified in this study, and will be explained in the following sections.

6 MODELS DETECTED FOR THE FIRST PROBLEM

We will firstly focus on the resolution of the problem of estimating the number of people that fit in the school playground (problem A).

Not all the individual proposals included a resolution scheme that enabled the required estimation to be made, but we however observed that the teamwork yielded suitable work plans for all groups. The students' proposals described the situation by means of mathematical concepts and their relationship to the studied reality as well as the procedures required to reach a solution, which means they modelled the problem following Lesh & Harel's (2003) definition.

It is worth noting that the high school playground has a rectangular plan view.

In the following we present the mathematical models created by the different teams to estimate the amount of people in the high school playground, which has a rectangular plan view.

Four of the teams used a strategy based on the idea of population density. The students measured the length and width of the playground in order to obtain its surface area. On the other hand they carried out experiments to determine the number of people that would fit on a small surface area. Using the experimental data, they obtained a value for the density of students that would take up one square metre and then multiplied it by the surface area of the playground in square metres.

One of the teams based its strategy on the iteration of a reference point. The students measured the length and width of the playground in order to calculate its surface area and carried out experiments to determine the area a single person would occupy. Using this value, they divided the total surface area of the playground in square metres by the surface taken up by one person. This process is equivalent to the iteration of a reference point, which is a length estimation strategy which consists of mentally counting the number of times and object (reference point) may be placed on top of the object to be measured (Joram, Gabriele, Bertheau, Gelman & Subrahmanyam, 2005).

One of the teams based its strategy on a grid distribution. The students in the team lined up one after the other. In order to move the line forward, the last person in the line would advance to the position in front of the first person in the line. While moving forward they counted how many people would be needed to occupy the length and width of the playground. To find out how many people would fit in the playground, they multiplied the two experimentally calculated values. This model responds to a similar idea to the product rule used to obtain the surface area of a rectangle.

We would like to stress that the previous three models are based on a type of resolution which establishes different sub problems that must be solved separately: What's the surface area of the playground? How many people fit in a square metre? Or: how many people can we line up along the length of the playground? This way of proceeding corresponds to Ärlebäck's (2009) definition of Fermi problems.

Once they completed the activity, the students reported their results, as well as the methodology used to obtain them. This idea-sharing session succeeded in getting the students to compare their methods and adopt the models their classmates had used. They also discussed their results, since the values obtained ranged between 1200 and 2200 people for a playground of 350 square metres. The students accepted the idea that there couldn't be just one single correct result, but a suitable interval including all possible solutions.

Afterwards the results were compared with capacity data from concert venues and the students realised their density values were rather high. This triggered discussions in which they tried to clarify the points in which they disagreed (surface area occupied by an adult versus that occupied by a teenager, comfort, safety rules). This allowed the students to connect the model used and the decisions taken with the reality being studied.

7 ADJUSTMENT OF MODELS TO THE FOLLOWING PROBLEMS

In the second session the students solved the remaining 4 problems. This time they weren't allowed to go to the locations referred to in the problems to take any measurements, and they therefore decided to search for the required information on the internet. Given that the formulations of the problems contextualized the estimation in public spaces, the students were faced with the difficulty of deciding how to adapt the models they had built for the previous problem to the new problems set. Given that the data we are working with is the students' output, this study cannot analyze the cyclical process of modelling for a specific problem. However, we can study the modifications made to adapt certain models for their use on different problems.

The first notable fact is that the teams generally used the population density model for the new problems, possibly because they considered this model more versatile and adaptable to any situation. Only two of the teams used the iteration of a unit, the space taken up by a tree, in order to solve the problem on Central Park. They considered that the density of trees expressed as units by square metre wasn't a manageable number.

After analysing the resolutions presented, we detected several variations in the models used by the students.

The students identified unavailable spaces when devising the resolution strategies for problems B2, B3 (squares in an urban centre) and B4 (trees in Central Park). Contrarily to the case of the school playground, it's not possible to cover the whole extent of a public space, since some of it taken up by street furniture or roadways. Considering this constraint, the students looked for an aerial photograph of the areas to be studied in Google Maps, delimited inoperative areas and marked them. Then they made some measurements with a tool from Google Maps in order to calculate surface areas.

The students identified spaces with different densities when embarking on the resolution of the problem which refers to the Palau St. Jordi, a large pavilion where concert attendants can either sit on the tiers or stand in the arena. They were unable to find an aerial photograph of the pavilion interior because it is indoors. They needed to approximate the surface area of each of the two parts by educated guesses. Once estimated the surface areas, they applied different population densities to them.

In problems B1 (population density of the steps), B2, B3 (population density in a demonstration) and B4 (density of trees) the students realised that the density of the objects may vary according to the circumstances. When estimating the population density of trees in Central Park, the students collected information regarding types of trees and their ages. They also determined that the density of people seated on one of the tiers in Palau St. Jordi is different from that of the people standing in the concert. They also distinguished between the population density in a concert from that of a demonstration. When the students completed their estimations using methods they had created themselves, the activity ended up in a discussion on the validity of the results obtained. They used information from different sources on the web (Wikipedia and several online newspapers) in order to contrast the results and even discarded some of the statements made by the media. Therefore, the students validated their results and established direct links between the studied reality and the models they had created and adapted to different situations.

8 CONCLUSIONS

Based on the collected material and data analysis, we can state that the pupils in this study solved the problems presented to them as Fermi problems, by establishing small sub problems they solved separately by means of calculations or estimations. For the problem on the number of people that would fit in the playground, the students' proposals included different types of models. This suggests that working with different Fermi problems may help and encourage students to generate a wide range of strategies and models, which agrees with Ärlebäck's (2011) statements on the possibilities offered by Fermi problems.

Using an initial problem to be worked on at school followed by list of similar problems that require researching information in external sources has allowed the students to discuss how the previously created models adapted to the different problems (Ärlebäck, 2011). This discussion has driven the students to generate new models which adapt to the new situations to estimate the amounts required in each of them.

When working with problems which pose new difficulties, the students adapted their models to the represented reality, creating new and more complex models. Since the work was carried out in teams, we ensured that all the students in class took part in the discussion on the different key aspects of decision-making for generating the models used. This allows the students to adopt these models and include them in their knowledge base (Schoenfeld, 1992). Therefore, the proposed sequence of problems has led to a higher level of understanding of the methods and concepts used; connecting those to different situations which have similar approaches but differs in some content details. It is worth noting that students started their project using the models suggested by their own team but easily adopted ideas proposed by other teams, which implies that the different idea-sharing sessions of methods and results were crucial to the whole process.

Finally, we observe that students are able to compare some of their results with information gathered from different sources. This comparison provides information which they may include when elaborating future models, such as the need to decrease the population density for safety reasons. On the other hand, some of the data found on the web did not agree with some of the results the students had obtained using their most perfected models, accepted by the whole class, which lead them to question the truthfulness of this information. Manifestly, the process reached its most relevant point when the students realised that their own analysis of a situation may disprove information provided by the media. In conclusion, by means of these resolution processes the students have given meaning to the concepts worked on in class and compared their output with real information, as described by Jurdak (2006).

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