CHARACTERISING TRIANGLE CONGRUENCY IN LOWER SECONDARY SCHOOL: THE CASE OF JAPAN

Keith Jones* & Taro Fujita**

*University of Southampton, UK; ** University of Plymouth, UK

Congruence, and triangle congruence in particular, is generally taken to be a key topic in school geometry. This is because the three conditions of congruent triangles are very useful in proving geometrical theorems and also because triangle congruency leads on to the idea of mathematical similarity via similar triangles. Despite the centrality of congruence, and of congruent triangles, there appears to be virtually no research on the topic. In this paper we report on what characterises the approach to triangle congruency in textbooks used in lower secondary school in Japan, specifically in Grade 8 (students aged 13-14). We found that practical tasks contained rich conceptions (measuring, transformations, etc.) whereas proof problems expected students to utilise corresponding parts and known facts.

Keyword Geometry, conceptions of congruency, textbooks, Japanese lower secondary schools

INTRODUCTION

The thirteen books of Euclid, compiled over 2000 years ago, are universally acknowledged as a ground-breaking example of the use of the deductive method in proving a collection of mathematical theorems. Covering mostly plane and solid geometry, but also some elementary number theory and algebraic methods, the books subsequently became what Boyer (1991, p. 119) has called as "the most influential textbook of all times". In proving many of the geometrical theorems, the books of Euclid rely on the three conditions for triangle congruency: the side-angle-side (SAS) condition (proved as Proposition 4), the side-side-side (SSS) condition (proved as Proposition 8), and the side and two angles (SAA) condition (proved as Proposition 26). These three conditions for congruency (SAS, SSS, SAA) are subsequently used in the books of Euclid to prove many more propositions.

The power of the three conditions for triangle congruency for proofs in geometry, exemplified by the books of Euclid, together with the link from triangle congruency to the even more powerful mathematical topic of similarity (via the idea of similar triangles), means that congruency constitutes a key topic in school geometry. This important position is exemplified by the TIMSS (Trends in International Mathematics and Science Study) curriculum framework which captures the geometry curriculum within two components; namely "position, visualization, and shape" and "symmetry, congruence and similarity" (Robitaille et al., 1993, appendix C).

Despite the key position in the geometry curriculum of congruence in general, and of congruent triangles in particular, there is, as far as we have been able to ascertain, virtually no research on the topic (at least, virtually none published in English. In

Japan, for example, there are many papers written by practitioners and teachers which provide useful ideas for daily lessons (e.g. Moriya, et al., 2005), but Shimizu (1979) discussed mathematical and pedagogical values of the congruency conditions.

One article on the topic is that by González and Herbst (2009). Utilising the notion of 'conception' from Balacheff (see Balacheff and Gaudin, 2003, 2004), derived from an epistemological position that "A knowing is characterised as the state of dynamical equilibrium of an action/feedback loop between a subject and a milieu" (p. 8). Referring to this framework, González and Herbst define four "conceptions of congruency" (ibid p. 155) and show how when using dynamic geometry software "There was evidence for students' learning in their shift from a visual perception conception of congruency to a measure-preserving conception of congruency" (ibid p. 179).

This raises the issue of how triangle congruency is characterised in other approaches to the topic. As part of a larger-scale project examining how geometry is presented in school mathematics textbooks (see, for example, Fujita & Jones, in preparation, and Jones & Fujita, in preparation), in this paper we report on a component of the project that addressed the following research questions: what characterises the approach to triangle congruency in textbooks used in lower secondary school in Japan? In what follows, we first expand on our focus on the geometry curriculum and school mathematics textbooks. Next, we outline the theoretical framework of four "conceptions of congruency" as introduced by González and Herbst (2009, p. 155). Following an explanation of the context for our study, and an outline of our methodology, we then provide our analysis of sample chapters from a Grade 8 textbook commonly used in Japan for students aged 13-14.

THE GEOMETRY CURRICULUM AND SCHOOL TEXTBOOKS

While geometry is an indisputably-important component of the school mathematics curriculum, arguments about the structure of the geometry curriculum have been going on for at least one hundred years (see, for example, Sinclair 2008; Usiskin 1987). As one well-known curriculum team from the early 1970s commented "Of all the decisions one must make in a curriculum development project with respect to choice of content, usually the most controversial and the least defensible is the decision about geometry" (The Chicago School Mathematics Project, 1971, p. 281). Such controversy means that the geometry provides an intriguing focus for studying how the mathematics curriculum is enacted through the medium of school mathematics textbooks.

In terms of textbooks, evidence from TIMSS (such as that documented in Valverde et al. 2002) has revealed the extent to which such textbooks provide the link between the *intended curriculum* that is contained within National Curriculum documentation and the *attained curriculum* that is learnt by students. This is done through the *implemented curriculum* that is taught by teachers. Textbooks lay out what Foxman (1999) has called the 'potentially-implemented' mathematics curriculum; that is, the curriculum that the teacher might implement in actual classroom practice. According to Thompson et al. (2012, p. 254) textbooks are a “particularly critical link between the intended and attained curriculum in school mathematics” because “they help teachers identify content to be taught, instructional strategies appropriate for a particular age level, and possible assignments to be made for reinforcing classroom activities”. It is how triangle congruency is enacted in school geometry textbooks in Japan that is the focus of what we present in this paper. It is to that context that we turn next.

TRIANGLE CONGRUENCY IN SCHOOL MATHEMATICS IN JAPAN

The specification of the mathematics curriculum for Japan is given in the ‘Course of Study’ (MEXT, 2008). Mathematical content is divided into ‘Numbers and Algebraic Expressions’, ‘Functions’, ‘Geometrical Figures’ and ‘Making Use of Data’. Our focus is ‘Geometrical Figures’ in Grade 8; Table 1 gives the detail of this topic.

Table 1 ‘Geometrical figures’ in course of study for grade 8

(1) Through activities like observation, manipulation and experimentation, to be able to find out the properties of basic plane figures and verify them based on the properties of parallel lines.
(a) To understand the properties of parallel lines and angles and basing on it, to verify and explain the properties of geometrical figures.
(b) To know how to find out the properties of angles of polygons based on the properties of parallel lines and angles of triangle.
(2) To understand the congruence of geometrical figures and deepen the way of viewing geometrical figures, to verify the properties of geometrical figures based on the facts like the conditions for congruence of triangles, and to foster the ability to think and represent logically.
(a) To understand the meaning of congruence of plane figures and the conditions for congruence of triangles.
(b) To understand the necessity, meaning and methods of proof.
(c) To verify logically the basic properties of triangles and parallelograms based on the facts like the conditions for congruence of triangles, and to find out new properties by reading proofs of the properties of geometrical figures.

From Table 1 we see that understanding of congruent figures is one of the main objectives. Also we can see that the conditions of congruent triangles are expected to be used in verifying properties of geometrical figures. Our interest is what characterizes the approach to congruency presented in textbooks as this mediates between the intended curriculum and classroom practice.

ANALYTIC FRAMEWORK AND METHOD

As we note above, in this paper we are following Balacheff and Gaudin, (2003) and González and Herbst (2009, p. 154), taking a ‘conception’ as being “the interaction between the cognizant subject and the milieu – those features of the environment that relate to the knowledge at stake”. In this framework, a conception comprises the following quadruplet (P, R, L, ∑) (Balacheff and Gaudin, 2003; González and Herbst, 2009):

- P: a set of problems or tasks in which the conception is operational,
- R: a set of operations that the agent could use to solve problems in that set,
- L: a representation system within which those problems are posed and their solution expressed, and
- ∑: a control structure (for example, a set of statements accepted as true)

In their paper, González and Herbst (2009, pp. 155-156) propose the following four conceptions of congruency.

- The *perceptual conception of congruency* (PERC) "relies on visual perception to control the correctness of a solution to the problem of determining if two objects (or more) are congruent".
- The *measure-preserving conception of congruency* (MeaP) "describes the sphere of practice in which a student establishes that two objects (e.g. segments or angles) are congruent by way of checking that they have the same measure (as attested by a measurement instrument)".
- The *correspondence conception of congruency* (CORR) is such that "two objects (segments or angles) are congruent if they are corresponding parts in two triangles that are known to be congruent".
- The *transformation conception of congruency* (TRANS) "establishes that two objects are congruent if there is a geometric transformation, mapping one to the other, which preserves metric invariants".

Learners (subjects) interact with textbooks to access, examine, and develop their mathematical knowledge, and our intention is to characterise textbooks in terms of the quadruplet. In terms of textbooks used in Japan, we selected *Mathematics 2* as this is widely used in Japanese lower secondary schools (Grades 7-9). First we identified the 34 intended geometry lessons in the Grade 8 textbook. The next step in our analysis procedure was derived from the work of Valverde et al. (2002) and consisted of the following steps: division of each lesson into 'blocks'; coding of each 'block' in terms of 'content' and 'performance expectation'. By using this method, we identified lessons and activities related to congruent triangles: this revealed that conditions of congruent triangles are introduced in the 8^{th} lesson, and after this lesson, a further 22 lessons refer to congruent triangles. Of these 23 lessons in total that refer to congruent triangles, 19 include some proving opportunities (for additional analysis, see Fujita and Jones, in preparation; Jones and Fujita, in preparation).

Of the 23 lessons which are related to congruent triangles, we then closely look at eight lessons in Chapter 4 Section 2 Congruent figures. The lesson details are given in Table 2. In these lessons, we decided to analyse activities and problems from the lessons 8, 9, 10, 11 and 13. We excluded Lesson 12 because this lesson is to consolidate students' knowledge and understanding of elements of proofs such as suppositions, conclusions and so on. Lesson 14 and 15 are excluded because the nature of problems are not radically different from the problems in lesson 13. In summary, at this first stage of analysis, we analyse 11 activities and problems.

Table 2 Lesson details in chapter 4 section 2 congruent figures

Lesson No.	Description
8	Introduction of congruent figures with 1 activity & 2 problems
9	Introduction of conditions of congruent triangles 1 activity & 1 problems
10	How to use the conditions of congruent triangles with 3 problems
11	Proofs using the conditions of congruent triangles with 1 activity
12	How to write proofs with 1 activity and 2 problems
13	Two proof problems with section summary
14 & 15	Section exercise (Mainly proof problems)

Following the approach of González and Herbst, our next steps were to undertake a priori analysis. In our analysis process, in addition to Balacheff and Gaudin (2003), we refer to Mesa's work (2004) to identify the quadruplet of the congruency tasks in the textbook (p. 262).

- to use the quadruplet (Problems; Operations; Representation system; Control structure) to characterise the selected lessons in the G8 textbook;
- to use the information from our analysis to characterise the approach to triangle congruency utilised in the textbook

As an example of how we analysed the textbook lessons, let us take 'Problem 1' in Lesson 9 (noting that at this stage the conditions of congruent triangles, SSS, SAS, ASA, have yet to be formally introduced); "Will each of the following conditions result in only one possible triangle? Investigate by actually drawing triangles. (1) A triangle with three sides: 4cm, 5cm, 6cm. (2) A triangle with a side of 6cm and two angles measuring 30º and 45º" We coded the above problem in terms of the quadruplet (P, R, L, $\sum$) as set out in Table 3. For 'P', this problem focuses on the number of triangles (whether only one or more than one) by using the given data. Students are expected to *operate* various actions to tackle this problem, e.g. drawing various triangles by using given data, observing whether two triangles are identical and so on. Also, students might measure angles, they might overlap triangles to verify whether two triangles made by given data are identical (congruent) or not. *Representations* mediate operations and control systems, and in this problem they are mainly graphical, e.g. the constructed diagrams as the medium of solutions, measurement tools as the medium for lengths of sides and sizes of angles, numerical values of angles and sides as the medium of solutions and so on. The operations are validated by *control structure*, i.e. overlapping two figures are validated if they are just overlapped, two triangles can be the same if all sides and angles measured are equal and so on. We attempted to identify all possible activities to verify operations.

Table 3 Conception of congruency in lesson 9

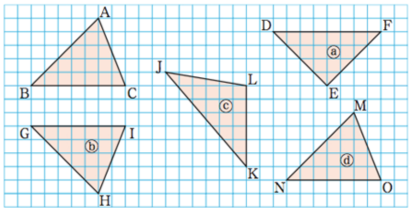
P: Problem	9Pa: The use of given data.
R: Operation	9Ra: To measure given data by using measurement tools. 9Rb: To draw triangles by using given data. 9Rc: To observe the figure resulting from movement. This involves either physical or mental manipulations. 9Rd: To measure sides and angles of triangles. 9Re: To overlap two constructed triangles. 9Rf: To observe appearance of triangles.
L: Representation system	9La: The constructed diagram is the medium for the solution of the problem. 9Lb: Measurement tools are the medium for lengths of sides and sizes of angles. 9Lc: Numerical values of angles and sides.
∑: Control structure	9∑a: If two constructed triangles look the same/different 9∑b: If all sides and angles measured are the same then two triangles are congruent. 9∑c: If two triangles are just overlapped then these two triangles are congruent.

FINDINGS

In what follows, in addition to the analysis of Lesson 9 Problem 1, we focus on analysing Lesson 8 and Lesson 13. Lesson 8 is the first lesson in which students in Grade 8 study congruent figures. The lesson starts from a practical activity and the result of our coding is given in Table 4. The first task in the lesson (as shown in Table 4) can be completed in several ways, either physically (e.g. moving triangles, measuring sides and angle, and so on) and/or mentally. Some students might relay on visual appearance to judge whether two particular triangles are congruent or not. Also squared paper can both operate various actions as well as mediating lengths of sides. At this stage, students have not learnt the conditions of congruent triangles; they are likely to rely on visual judgement as well as measurement data.

Following this lesson, the conditions of congruent triangles are formally introduced in Lesson 9; see our analysis of lesson 9 in Table 3. The next lesson, Lesson 10, starts from the following narrative “It can be determined if two triangles are congruent by checking to see if they satisfy any of the congruence conditions without having to check to see if they can completely overlap”. In this way, after Lesson 10 the conceptions of congruency intended in the activities and problems become more symbolic than graphical, i.e. students are expected to use already known facts and symbols to express congruency.

Table 4 Conception of congruency in lesson 8

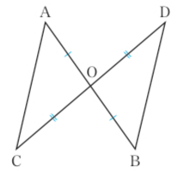
P: Problem	Q In the diagram below, which triangles can be moved to overlap completely with △ABC?  8Pa: To use given triangles.
R: Operation	8Ra: To move (rotating and translating) triangles and overlap them. 8Rb: To observe the figure resulting from movement. This involves either physical or mental manipulations. 8Rc: To measure lengths of sides and sizes of angles of each triangle. This involves either physical or mental counting for length as squared paper is used. 8Rd: To observe appearance of triangles.
L: Representation system	8La: Actual paper shapes are the medium for the presentation of the problem, but some simple cases can be done mentally by diagrams on paper. 8Lb: Squared papers are the medium for measuring lengths of sides. 8Lc: Measurement tools are the medium for lengths of sides and sizes of angles. 8Ld: Numerical values of angles and sides.
∑: Control structure	8∑a: If two triangles are completely overlapped then these two triangles are congruent (no specific functions or vectors are mentioned) 8∑b: If all sides and angles measured are the same then two triangles are congruent 8∑c: If two constructed triangles look the same/different

After Lesson 10, students are expected to use the conditions to prove various statements. For example, in Lesson 13, students are required to undertake reasoning problems involving two steps, as illustrated in Table 5. As might be expected, this problem needs more symbolic representation to mediate operations and control structure, as shown in the Table.

When we compare our findings to González and Herbst’s four conceptions of congruency, the following can be observed that conceptions identified in activities and problems in Lessons 8 and 9 are characterizes as ‘PERC’, ‘MeaP’, and ‘TRANS’. For example, in Lesson 8,

- PERC (P, R, L, ∑) as (‘8Pa: To use given triangles’, ‘8Rd: To observe appearance of triangles’, ‘8La: Actual paper shapes are the medium for the presentation of the problem’, ‘8∑c: If two constructed triangles look the same/different)
- MeaP (P, R, L, ∑) as (‘8Pa: To use given triangles’, ‘8Rc: To measure lengths of sides and sizes of angles of each triangle’, ‘8Lc: Measurement tools are the medium for lengths of sides and sizes of angles’, ‘8∑b: If all sides and angles measured are the same then two triangles are congruent’)
- TRANS (P, R, L, ∑) as (‘8Pa: To use given triangles’, ‘8Ra: To move triangles and overlap them’, 8La: Actual paper shapes are the medium for the presentation of the problem, ‘8∑a: If two triangles are completely overlapped then these two triangles are congruent)

Table 5 Conception of congruency in lesson 13

P: Problem	☐ Check 2 In the diagram shown on the right, if O is the midpoint of line segments AB and CD, then ∠OAC = ∠OBD.  13Pa: To use the conditions of congruent triangles.
R: Operation	13Ra: To identify what assumptions and conclusions are. 13Rb: To identify congruent triangles. 13Rc: To apply known facts to identify equal angles and sides. 13Rd: To apply the conditions of congruent triangles. 13Re: To deduce and identify equal angles.
L: Representation system	13La: The diagram is the medium for the presentation of the 13Lb: problem. 13Lc: The symbols are the registers of equal sides and angles. 13Ld: Already known fact such as the conditions and properties of congruent triangles mediate for the solution and reasoning.
∑: Control structure	13∑a: If we can find three equal components of triangles (SSS, ASA, SAS). 13∑b: If one of the conditions of congruent triangles is applied to two triangles. 13∑c: If two triangles are congruent, then the corresponding sides and angles are equal.

After Lesson 10, activities and problems (mainly proofs) are likely to be characterized as ‘CORR’, because of the facts and theorems, students are informed that corresponding parts are the same and therefore the two triangles are congruent.

Table 6 summarises the characterizations of congruency lessons in Japanese G8 textbook

DISCUSION AND CONCLUSION

Our aim of this paper is to characterise the approach to triangle congruency in textbooks in Japan. By using congruency conceptions as our analytical framework, we identify various intended conceptions in lessons related to congruency in textbooks. What we also found was practical tasks contain rich conceptions whereas proof problems expect students to have CORR as the main conception to solve problems. It seems that Japanese textbook we analysed assumes by designing the lesson progression, there will be conception changes from MeaP or PERC to CORR. An issue is that it is still uncertain, for example, that students who have just finished Lesson 10 fully develop CORR after several practical activities and a statement "It can be determined if 2 triangles are congruent by checking to see if they satisfy any of the congruence conditions without having to check to see if they can completely overlap."

In fact, Japanese G8 still struggle to solve geometrical problems, as exemplified in recent Japanese National survey result, which shows that about 40% of students are not sure why congruent triangles should be used to deduce a conclusion. As it is stated, conceptions are observed within "the interaction between the cognizant subject and the milieu". When we consider this, we should not simply conclude that these students are not capable to do proofs, but they might be still utilizing their MeaP or PERC when they face these proof problems of MeaP or PERC instead of utilising control structure in CORR '∑: If two triangles are congruent, then the corresponding sides and angles are equal', as these conceptions are certainly used lessons they faced prior to proof problems.

The conceptions we focused in this paper are those in textbooks, i.e. intended conceptions. Our next task is to characterise actual students' conceptions when they interact with various congruency problems. Also it is necessary to design learning progressions and activities which enable students to develop rich conceptions of congruency as well as how to control their conceptions when they face various problem situations.

Table 6 Conceptions of congruency in grade 8 textbooks in Japan

Conception	P	R	L	∑
PERC	8Pa 9Pa	8Rd 9Rc, 9Rf	8La 9La	8∑c 9∑a
MeaP	8Pa 9Pa	8Rc 9Ra, 9Rb, 9R d	8La-8Ld 9La-9Lc	8∑b 9∑b
TRANS	8Pa 9Pa	8Ra-8Rb 9Rc	8La 9La	8∑a 9∑c
CORR	10Pa 11Pa 13Pa	10Ra-10Rd 11Ra-11Re 13Ra-13Re	10La-13Le 11La-11Lc 13La-13Ld	10∑a-10∑b 11∑a-11∑c 13∑a-13∑c

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