AN APPROACH TO THE CONSTRUCTION OF THE IDEA OF AREA WITH QUALITATIVE AND QUANTITATIVE ASPECTS

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Abstract: Both qualitative and quantities aspects must be included in the idea of area when it's calculated by formulae as well as when we refer to its conservation. We proposed two procedures to build the idea of area to our high school students. In the quantitative procedure, triangular areas were calculated by inscribed triangles into adequate rectangles and in the qualities one, we reconfiguring parallelograms cutting their diagonals. The students could make a link between these procedures and they were able to express in writing an idea of area that unifies both aspects in the cases of the triangle, the rectangle and the parallelogram.

INTRODUCTION

In the teaching of area two approaches are generally used. One that can be considered as formal which refers to the calculation of areas with formulae and another, informal, that emphasizes the conservation of area in figures of a different shape. Another way to represent this duality is through the processing generally used in these approaches; while the calculation of the area with formulae is static and is used in prototype figures, area conservation is used for dynamism and in the reconfiguration, conserving the area.

In every case, these approaches have the quantitative and qualitative aspects of the area supporting them, which are often treated as separate methods. The main goal of this research the observation of how is possible to close the gap between the quantitative and qualitative aspects of the area. We propose a dynamic procedure of the calculation of the area of triangles to consider the quantitative aspect and a finite procedure of reconfiguration of parallelograms to rectangles to go over the qualitative aspect with that in mind.

We affirm that the unit of area is what permits the unification of both procedures while interpreting one in function of the other, therefore building an idea of area with qualitative and quantitative aspects for the cases of triangles, rectangles and parallelograms.

THEORETICAL FRAMEWORK

According to Kordaki the conservation of area must be taken into account as much as the calculation in the concept of area. The author also states that: In the school context students are introduced early to the use of area formulae but the concept of conservation of area is overlooked (Kordaki, 2003, p. 178)

While the conservation of area is not dealt with in class regularly it is also not acquired spontaneously and justifying it requires a complex process. The research that we conducted at Kospentaris et al. (2011) reports the strategies used by high school students to justify problems of area conservation of geometric figures. The authors found that most of them have difficulties employing an adequate formal reasoning, which is not considered necessary for the justification of this conservation. However, the visualization has a considerable influence in the answers.

Working with geometric figures requires making use of mathematical visualization. The objective is that of showing the link between figural units such as points, straight lines, closed contours, etc. Mathematical visualization is focused on the organization of the relations and in the last instance in these figure, Duval (2003) considers that:

> In any geometric figure, they have always been able to distinguish several shapes that figural units have as possible representations. That means that even the simplest Euclidian figures (circle, triangle, square…) have been looked on as a configuration of several figural units and never only as one figural unit. This hiding one important difficulty: Different identifiable figural units in a figure seldom have the same number of dimensions... On other hand, a perceptual organization of the variables can be used to learn to see it ... This dimensional variability to recognize figural units of a figure becomes an essential phenomenon to be taken into account to articulate geometrical visualization with a mathematical language, either a simple description, an explanation or a deductive reasoning. (Duval, 2003, p. 55)

As for the conservation of area, students emphasize the shape of the figure rather than the amount of it, but this also depends of the shape of the figure, in particular Kospentaries et al. (2011) claim that:

> The particular type of the compared figures seen to have an effect on the understanding of the area invariance: Although students accept the conservation in parallelograms, they face difficulties in the case of triangles. (Kospentaries et al., 2011, pp. 106-107)

Acknowledging that the same area can be contained in different drawing shapes is necessary in order to build the idea of conservation of area. If we begin with a figure to build another of different shape, we can use the visual processing of reconfiguration as an adequate resource.

To make use of the figural characteristics of a geometric figure, we distinguish four types of apprehensions, Duval (1999). They are: perceptual, sequential, discursive and operative, we focus on two of them which are especially important for this research: perceptual and operative, in words of Deliyianni et al.:

> To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least alone of the other three. Particularly, perceptual apprehension refers to the recognition of shape in a plane or in depth. In fact, one´s perception about what the figures shows is determined by figural organization laws and pictorial cues… Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refer to the division of the whole given figure into parts (reconfiguration), the optic way is when one made the figure larger or narrower, or slant , while the place way refer to the position or orientation variation. Each

> of these different modifications can be performed mentally or physically, through various operations. (Deliyianni et al., 2011, pp. 598-599).

Although reconfigurations of the original figures allow the underlining of figural relations as a source that can be used in the solving of geometric problems, it is important to remark that not all reconfiguration is adequate to obtain the desired results, Padilla (1992) for which it is necessary to study beforehand the possible compositions and adequate decompositions to obtain the viable reconfigurations.

On the nature of the manipulative that we Hill use in our investigation we share the position of Godino (1998) who affirms that:

> It is important to recognize that the used of the manipulative materials (tangibles) develop symbolic functions and the textual and graphic means also are manipulative. The rules, multibase blocks, the fingers of the hand, the Geoboard, the cubes etc., are tangibles sources that pose problems, with ordinary language and artificial mathematical symbols. But we consider them more than means of expression, but as instruments for the mathematical work either professionally or in school, that is, they are semiotics tools. (Godino, 1998, 200-201)

The use of manipulation in the qualitative as a quantitative procedure proposed in this work, are considered as mediation semiotic tools' that contribute to the constructions of the idea of area with qualitative and quantitative aspects.

We consider that gestures are of the resources that should be utilized in the construction of knowledge and that these can be presented with other demonstrations of semiotic type like language or writing, Arzarello et al. (2008). In fact, we consider, like Edwards (2005) that in decisive situations gesticulation can emerge to state and to organize the ideas in question.

> Gestures can be seen as an important bridge between imagery and speech, and may be seen as a nexus bringing together action, imagery, memory, speech and mathematical problem solving (Edwards, 2005, p. 136)

Finally, the unit of area presents by itself a series of difficulties due to that, in general, it is considered by our students as something given and immutable; in fact, a frequent mistake among them the confusion of measuring units with the number one, which causes the variation in size of the measuring unit, which is seen to be compromised by this flawed association. To consider a measuring unit that permits us to quantify the objects. The task by itself should show them that we can only compare areas if the unit is the same one.

There are many students who claim that the area is the amount of measuring units that fit it, while acknowledging that different rectangles, with the same number of cells, can have the same area, although they are radically apparent, Acuña (2010). This idea can produce mistaken assumptions about area.

According to Galperin and Georgiev (1969), when emphasizing the individuality of the unit, the student develops a justifiable indifference set against the size of the unit.

In this paper, the term "measuring unit" is used exclusively to account to the contained area units in a given rectangle, so that, in order to compare areas of rectangles either the counting of said units or overlapping comparisons are to be used. Given that the construction of an idea of area with qualitative and quantitative aspects is our goal, we leave the pondering on the properties of the units and their relation with the area for another time.

METHOD

We worked with 10 students between the ages of 15 to 18 for four sessions, 90 minutes each, in a workshop designed for the construction of an idea more general of area. The professor assigned an assistant of videotaping and the investigator took part in it. Figure construction activities were drawn in pencil and paper and the reconfiguration of figures was achieved with cutouts and drawings. At the end of each session, group discussions were carried out, and then worked out by couples which were later asked to put down in paper what they had understood during the session.

Taking advantage of the fact that the students are used to accepting the conservation of area in rectangles, we initiated the quantitative procedure postulate as a basic strategy of calculation with a statement that read: "if a parallelogram is cut by a diagonal, then the two triangles that are formed by it have an equal area"

This affirmation was justified with some examples and constructions. In the qualitative procedure, we take advantage of the diagonals cutting through cut and transforming parallelograms with sharp or obtuse angles, some angles close to 90º. The change of the figural units (sides and diagonals) was resolved based on the construction of appropriate figures and by a calculation of difference and the construction and choice of adequate diagonals.

We put into operation two procedures that we call (quantitative and qualitative, which are described subsequently) in this research.

Quantitative Procedure

The conservation of area in this fashion was achieved by using wooden Geoboard and the computer program Geogebra. In the first case, while the posts complicated the counting of the area units, this instrument provided the students with a friendly environment. In the second instance, however, the posts were imperceptible which permitted the students to reach the solutions much more quickly.

Three inscribed triangle situations were worked on. First case: calculation of areas was direct. Second: one must subdivide the zones one wishes to calculate. Third, a duplicate zone appears.

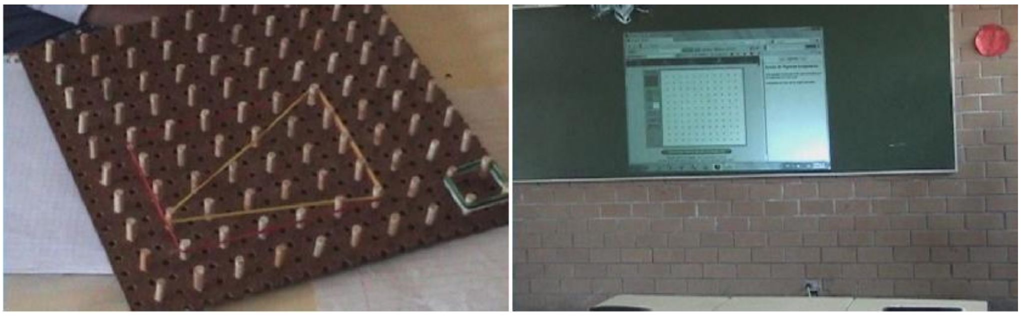

Fig. 1 Instruments of work: Geoboard and Geogebra

There was an incorrect general rule proposed by the students at the beginning of the practice: any triangle inscribed in a rectangle has half of its area of this.

This erroneous judgment was corrected calculating the areas involved.

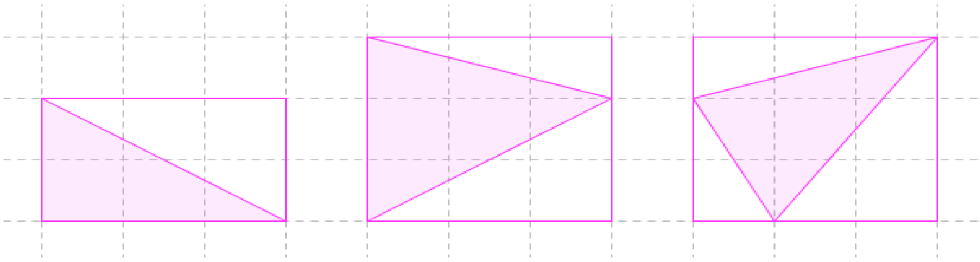

Fig. 2 Postulate basic strategies

The process of the unit of area was not exhaustive since didn't offer a range of dimensions nor we uses fractional measuring units, which are intentionally avoided in the triangle inscribed into a rectangle resource.

Qualitative procedure

The qualitative processing started with a difficult task for instance the below parallelogram, It was calculated by the Geoboard, which was undertaken through the reconfiguring using pencil paper and scissors with order of cut the diagonal to reconstruct another parallelogram of the same area, as it is seen in the figure 3

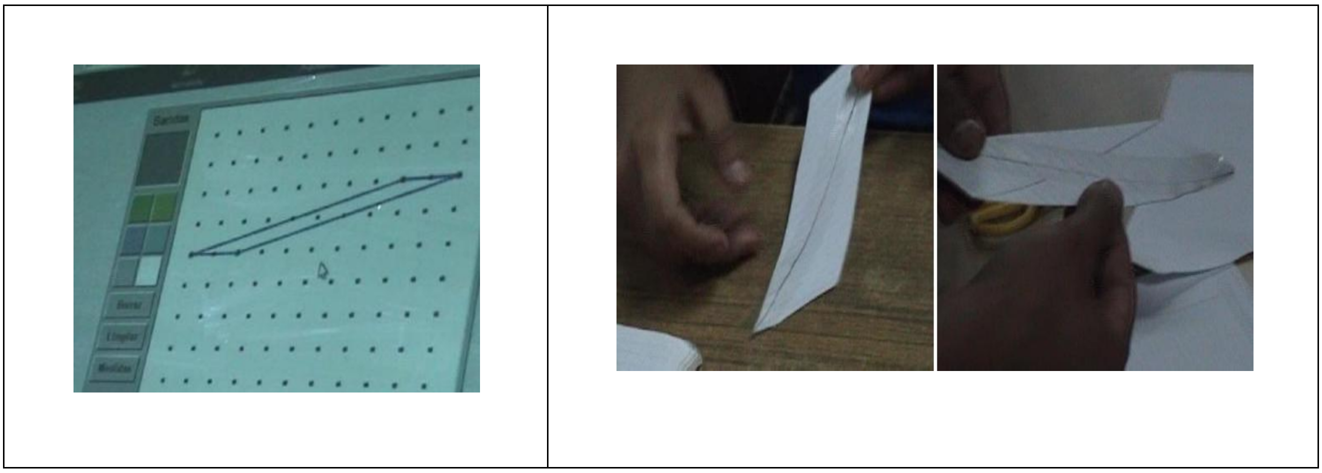

Fig. 3 Parallelogram cutting and reconfiguring by the diagonal

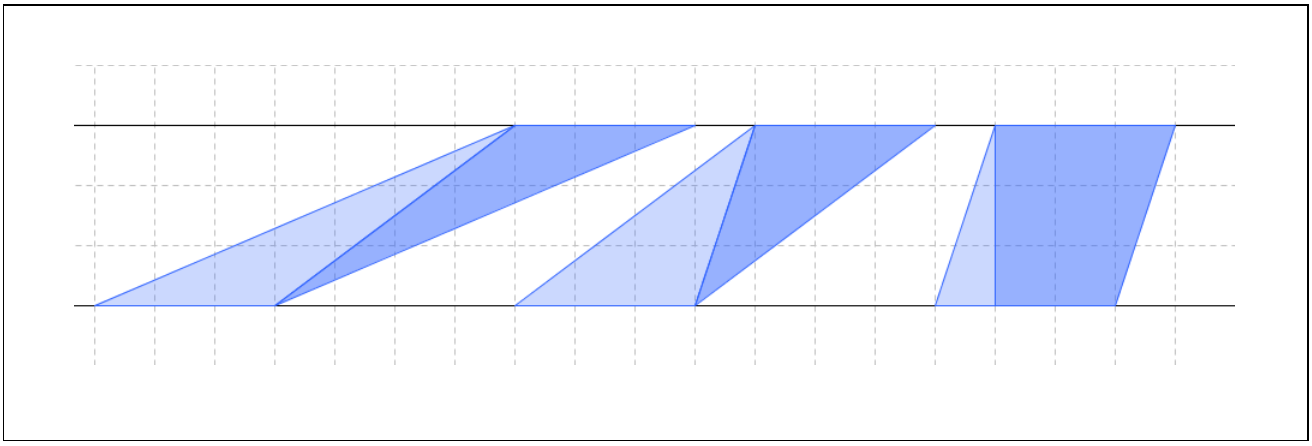

Fig. 4 Every parallelograms can be transformed into a rectangle

The qualitative procedure starts with the cutting of a parallelogram with an appropriate diagonal to transform it into a new parallelogram with internal angles closer 90 degrees. There is a moment when it is possible to make different, additional cuts. Not diagonally but perpendicularly in one of the sides, as is shown in the figure 4.

This procedure provided the students with both the certainty of the transformation, and a simple transformation strategy. A rectangle, which can be calculated in the quantitative procedure, is obtained. Once the parallelogram was transformed the area could be calculated counting units, an idea that took form in multiplying the base by the height.

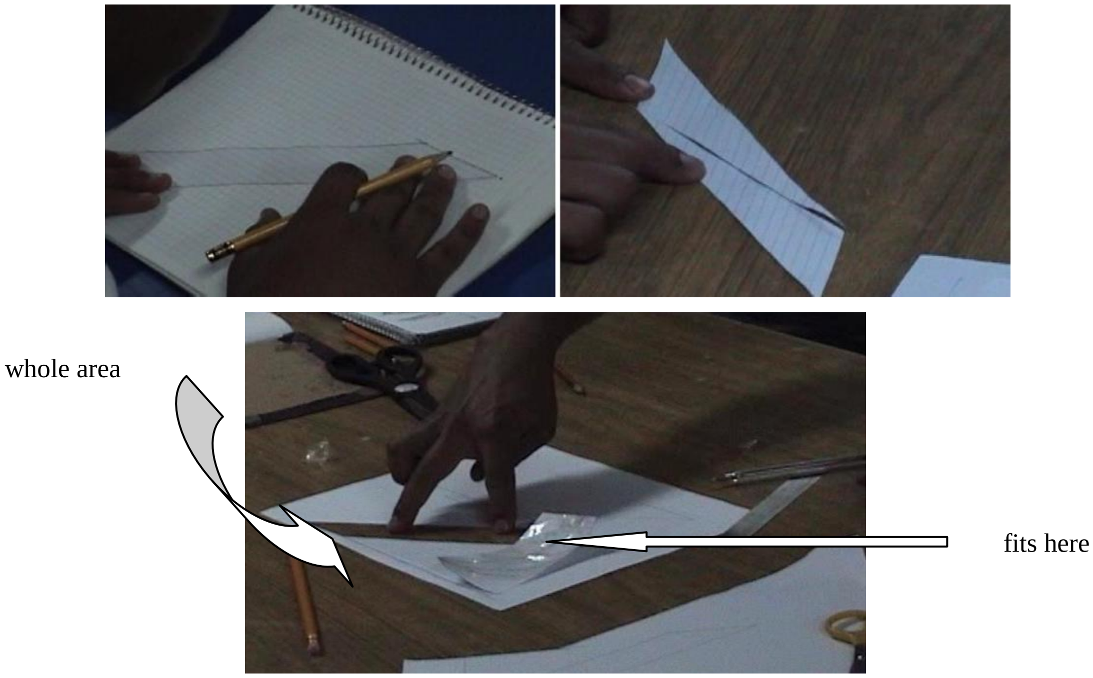

Fig. 5 Reconfiguring makes that students short the transformation by a single gesture; they joint all area in one zone

RESULTS

The basic quantitative strategies were of great help to the students. Once they accepted they could calculate the area of a triangle with an adequate rectangle, and then the problem is in choosing an adequate dissection to build the rectangles. This was resolved by the direct calculation of units that covered the rectangle, which was simple and viable.

The construction of the false generalization that said "every triangle inscribed in a rectangle has half the area of the rectangle" caused a moment of uncertainty which surpassed by means of a more careful exploration of the situation. This permitted us to lose the area that overlaps in the basic strategy number three, which too was calculated as a difference of rectangles.

Focusing the attention of students on figural relations and properties of the triangle and rectangles to calculate their areas was possible thanks to the availability of the initial strategies. To express the triangles´ area by units was product of the use of rectangles.

We observed, regarding the qualitative resources, that during the first time we cut and glued the triangles with adhesive tape to the paper parallelograms, the students expect that the accuracy in the cut and adhesion is what will permit the reconfiguring. The correctness was the base of the success of the task, was after all the way as they validated the procedure.

During the second time, accuracy was dismissed. The students accepted the viability without problem. The role that the base and height played was dutifully noted.

After several reconfigurations the students were ready to establish inductively that every parallelogram can be transformed into a rectangle of the same base and height as the original parallelogram. The general idea that establishes the relation between the original parallelogram and the rectangle was suggested in the explanation that they gave each other in the construction tasks.

The gestures of the students also agreed with the dynamic idea of the transformation, they moved their right hand from the right toward the left showing a movement for the upper base and another movement from top to the bottom to denote the height. At the end of the activity, the transformation was omitted completely and the attention was directed to the size of the base and height.

As we mentioned before, the area units played an important role in the means through which the students could refer to the qualitative and quantitative properties of area. About the idea of area with qualitative and quantitative aspects, as we mentioned before the unit of area played an important role of the meant through which the students could refer to this kind of properties.

We have displayed some examples produced by students where we can see details on the conservation of the area and the reconfiguration as well as drawings that show a dynamic approach to the calculation and that include conservation, and the counting of the units involved at the same time.

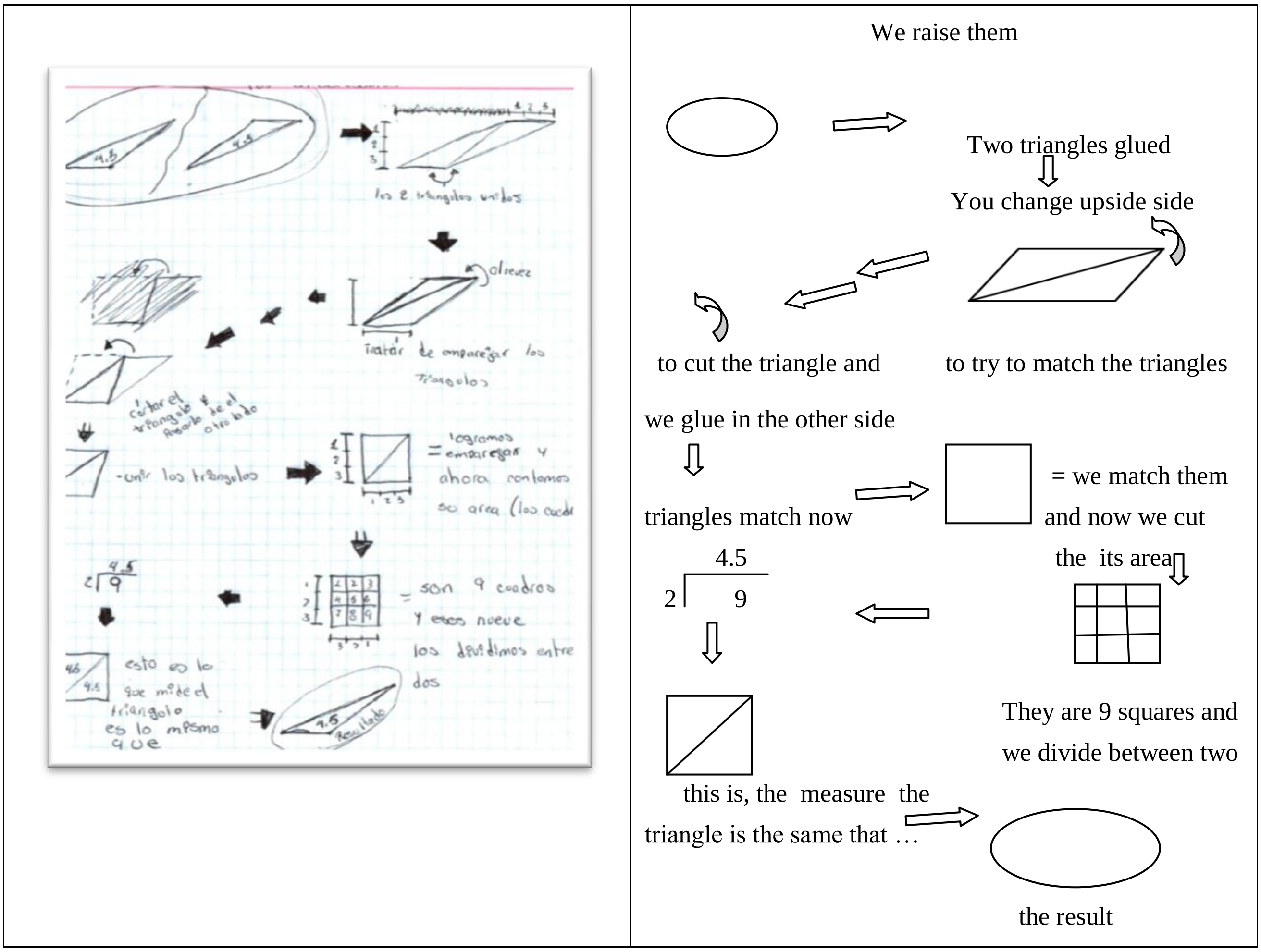

Fig. 6 Both procedures

The samples produced were influenced by three elements: parallelograms that are change dynamically to rectangles, triangles that are calculated with rectangles and marks that refer to measurements and counting.

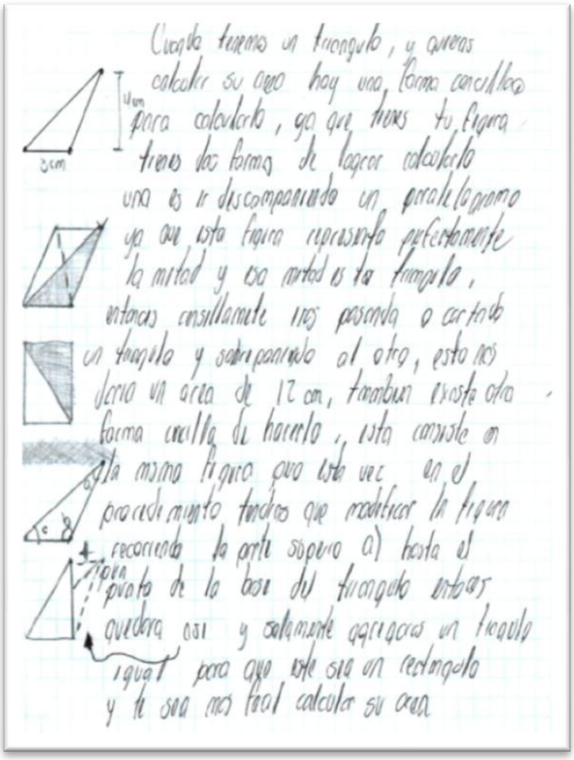	When we have a triangle, and we want calculate its area, there is an easy way to do. When you take the figure you have two ways to calculate. One way is decomposing one parallelogram since this figure representing perfectly the half and this half is the formulae, then easily we pass it and overlapping the other, this give us an area of 12 cm. There is other way to do this is about the same figure but in the procedure we have to modify the figure moving the top part a) until the base point, then ↖ it looks like this and you only add an equal triangle to complete a rectangle and it is more easy to calculate the area.

Fig. 7 Students writings

DISCUSSION

The procedures proposed to the students and the discussion, first in couples and later encouraged by the teacher with the entire group produced an idea of area with qualitative and quantitative aspects. This idea helped to build a bridge between the formulae and the conservation of area; it is not exhaustive but touches the credibility and believes of the students about what is the area. It is possible to design and to focus more and deeper situations to cover more geometrical figures in the future.

These procedures allowed us to focus on the figural units and its properties as a source of valid and information. They provided a dynamic treatment to calculate areas of triangles as well as rectangles and parallelograms and we believe to have achieved our goal.

We think that the journey for students to transform their idea of the area into a both qualitative and quantitative object is long and difficult. In this article we showed a suggestion of a treatment which has good possibilities of being effective. Nevertheless, we have to research about the stability of the students' perception about the area in other cognitive situations.

REFERENCES

Acuña, C., (2010). Las funciones visuales y epistémicas de los dibujos, *Numero especial de la Revista Latinoamericana de Investigación en Matemática Educativa RELIME* 13,4, pp. 115-128.

Arzarello, F., Paola, D., Robutti, O. & Sabena, C., (2008). Gestures as semiotic resources in the mathematics classroom, *Educational Studies in Mathematics*, 70, 97-109.

Duval, R., (1999). Représentation, Vision and Visualization: Cognitive Functions in Mathematics Thinking. Basic Issues for Learning. *Proceedings of the 21st the American Chapter of the Psychology of Mathematics Education,* PME-NA Annual Meeting, México, 1, 3-26.

Duval, R., (2003). "Voir" en mathématiques, En Filloy. (Ed.), *Matemática Educativa. Aspectos de la investigación actual*. Centro de Investigación y Estudios Avanzados del IPN, Fondo de Cultura Económica, México, 41-76.

Deliyianni, E., Gagatsis, A., Kalogirou, P. & Kusniak A., (2011), Towards comprehensive theoretical model of students´ geometrical figure understanding and its relation with proof, *Proceedings VII of European Research in Mathematics Education CERME 7,* 598-607, on line: http://www.cerme7.univ.rzeszow.pl/WG/4/WG4_deliyianni.pdf

Edwards, L., (2005). The role of the gestures in mathematical discourse: remembering and problem solving, *Proceedings of the 29th annual meeting of Psychology of Mathematics Education, RF02,* PME29, 1,135- 138.

Galperin, P. & Georgiev, L., (1969). The formation of elementary Mathematical Notions, In J. Kilpatrick & Wirszup (Eds.), *Soviet Studies in the Psychology of learning and teaching Mathematics, Vol. 1:The Learning of Mathematical Concepts.* Chicago, USA: University of Chicago, pp. 189-216.

Godino, J., (1998). Uso de material tangible y gráfico-textual en el estudio de las matemáticas, *Actas do ProfMat 98,* A. M. Machado y cols. (Ed.) Associaçao de Professores de Matemática, 117-124.

Kordaki, M., (2003). The effects of tools of a computer microworld on students´ strategies regarding the concept of conservation of area, Educational Studies in Mathematics 52, 177-209.

Kospentaris, G., Spyrou, P., & Lappas, D., (2011). Exploring students' strategies in area conservation geometrical tasks, *Educational Studies in Mathematics,* 77, 105-127.

Padilla, V., (1992). *L´influence d´une acquisition de traitements purement figuraux pour l´apprentissage des mathématiques*, Thèse de Docteur, L´institut de Recherche Mathématique Avancée, Université Louis Pasteur et C. N.R.S.