CONCEPTUAL CHALLENGES FOR UNDERSTANDING THE EQUIVALENCE OF EXPRESSIONS – A CASE STUDY

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Whereas students' conceptual understanding of variables and equations has often been investigated, little is known about students' pathways to understanding equalities, i.e. the equivalence of expressions as generational instead of transformational activities. The case study reconstructs two main conceptual challenges that must be overcome on the pathways to conceptual understanding for the equivalence of expressions: (1) limited degrees of generality for variables and geometric figures, and (2) operational versus relational perspectives on expressions.

THEORETICAL AND EMPIRICAL BACKGROUND

Operational and relational meanings of the equal sign

Students' limited understanding of algebraic equality has often been problematized in terms of the dichotomy between *relational and operational interpretations of the equal sign* (Kieran, 1981; McNeil & Alibali, 2005). Although many students only interpret the equal sign *operationally* as a prompt to calculate the value, algebraic thinking also necessitates the *relational* meaning as signifying symmetric structural relations between left and right side of the equal sign.

Whereas most research has focused on equations like $7x + 28 = x + 4$, the relational interpretation of the equal sign is not only addressed *(A) in algebraic equations*, but also in three other important aspects: *(B) arithmetical identities* like $7 \cdot (10+4) = 7 \cdot 10 + 7 \cdot 4$, *(C) equivalence of expressions* like $7 \cdot (x+4) = 7x + 7 \cdot 4$, being generalized from (B), and *(D) contextually bound identities* like "Right triangles with hypotenuse c and legs a, b satisfy $a^2+b^2=c^2$" (cf. Prediger, 2010 for these aspects). Aspect (B) refers to arithmetic; (A), (C) and (D) to algebra, but with different meanings of the variable: Whereas variables in equations (A) serve as unknown to be solved, variables in aspects (C) and (D) serve as generalized numbers (Usiskin, 1988, p. 17). Hence, generalization is crucial for understanding algebra. As students' understanding of aspect (C), equivalence of expressions, has much less been investigated and fostered than of equations (exceptions are Demby, 1997; Kieran & Sfard, 1999), we defined this aspect as the core subject of our design research project (cf. Prediger & Zwetzschler, 2013, for an overview).

In this paper, we present a small descriptive study within a larger design research project that focuses on the empirical specification of conceptual challenges that students encounter while developing conceptual understanding for the equivalence of expressions. We investigate these individual conceptions in a learning arrangement that promotes generational activities before transformational activities (Mason et al., 1985), as will be further explained in the next section.

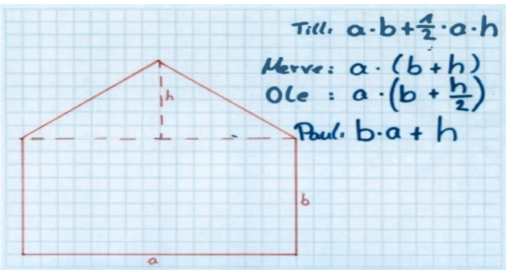
Three meanings for the equivalence of expressions

In our approach, we *generalize the operational-relational dichotomy from the equal sign to the equivalence of expressions*: How should students understand equivalences like a · b + 2 · b · h/2 = b · (a+h)? In line with the wrong priority attributed to operational meanings of the equal sign (Kieran, 1981; McNeil & Alibali, 2005) is the fact that many students (and some curricula) think about equivalence of expressions only in terms of *transformational activities*. But how to ground these transformation rules in conceptual understanding? Mathematically, they can be derived from the basic arithmetical laws (like commutativity and distributivity). But as Demby (1997) has pointed out, the general deduction from arithmetic to algebra is too complex and abstract for many learners (cf. Lee & Wheeler, 1989). That is why a learning arrangement that fosters the development of conceptual understanding of equivalence of expressions should first involve its inclusion in *generational activities*, in which algebraic expressions are not only understood as a system of meaningless signs (being transformed according to arbitrary rules), but as pattern generalizers of arithmetical or geometrical pattern (Mason et al., 1985, p. 46 ff.; Kieran 2004, p. 23). Within these activities, a *relational* understanding of the equivalence is achieved by comparing expressions with respect to equivalence Kieran & Sfard, 1999) To sum up, three meanings of the equivalence of expressions that are to be acquired; first, (a) and (b), then (c): two expressions are equivalent, if…

(a) *description equivalence*: …, if they describe the same phenomenon (same geometric pattern, same situation, same function, …);
(b) *insertion equivalence*: …, if they have the same value for all inserted numbers;
(c) *transformation equivalence*: …, if they can be transformed into each other according to the transformation rules. (Malle, 1993; Prediger, 2009)

For (a), the description equivalence, Kieran & Sfard (1999) compare functions where the table representation immediately leads to (b), the insertion equivalence. Our Tasks in Figure 1 (Prediger et al., 2011) follow Mason et al. (2005) and Malle (1993) who refer the descriptions to be compared to areas of varying geometric shapes. In this paper, we reconstruct critical moments in the learning pathways towards description and insertion equivalence; the later completion by transformation equivalence is not treated here.

(I) Which students calculate the same area? And which of the *expressions* calculate the area of the given geometric shape correctly?



(II) Insert different numbers for the variables. Check which of the expressions are equivalent.

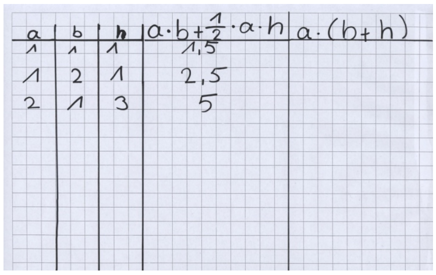


Fig. 1. Tasks (I) for experiencing description equivalence and (II) for insertion equivalence

Generalizing the *operational-relational dichotomy from the equal sign to expressions*, we developed Task I to prioritize relational perspectives on algebraic expressions against purely operational perspectives. That means, we do not emphasize the activity of *calculating* values of expressions, but of *formulating, interpreting and structuring* expressions while relating them to geometric shapes. However, our empirical analysis will show (in Section 3) that students still adopt other variants of operational perspective which produces conceptual challenges for their pathway to description equivalence.

Additionally, we will show that the *degree of generality* is a relevant source of difficulties: The insertion equivalence is quite natural for students for one *specific* insertion, namely the specific side lengths in the given geometric shapes; this interpretation of variables is known as "letter as object" (Küchemann, 1981). However, two expressions are only equivalent if they have the same value for *all* inserted numbers. We will show how a limited degree of generality (being transferred from geometry to algebra) forms a second conceptual challenge for understanding general insertion equivalence.

METHODOLOGY OF THE CASE STUDY

This study is embedded in a larger design research project (Prediger & Zwetzschler, 2013) that follows the methodology of Cobb & Gravemeijer (2006) with its dual aim of deepening the understanding of learning processes and designing learning arrangements. Therefore, it applies iterative cycles of (re)design and empirical investigation. Here, we concentrate on one step of empirical investigation with following research questions:

Q1 Which conceptions do students activate or develop in a learning arrangement designed to foster the conceptual understanding of description and insertion equivalence?

Q2 How do the individual conceptions of variables, expressions and geometric shapes influence the learning pathways? Where do conceptual challenges appear?

Data gathering in design experiments

The tasks presented in Table 1 were part of the teaching-learning arrangement used for twelve design experiments in laboratory settings (Komorek & Duit, 2004). A teacher worked with 12 x 2 students of grade 7 to 9 in German comprehensive secondary schools), for three to five sessions of 45 to 60 minutes, the presented task lasted 20 to 50 minutes. All experiments were videotaped and partly transcribed.

Data analysis: Vergnaud's analytical model of concepts- and theorems-in-action

For the interpretative analysis of individual conceptions (research question Q1), we operationalized "conceptions" by adapting the theoretical constructs concepts- and theorems-in-action from Vergnaud's theory of conceptual fields, as this theory offers "a fruitful and comprehensive framework for studying complex cognitive competences and activities and their development" (Vergnaud 1996, p. 219).

The *first step* of our analytic procedure allows us to reconstruct, for each of students' visible activity or utterance, the underlying operational invariants: *Theorems-in-action* are defined as "proposition that is held to be true by the individual subject for a certain range of situation" (Vergnaud 1996, p. 225). For adapting Vergnaud's construct to our specific needs, we symbolize theorems-in-action by <…> and always formulate the purpose and the means, e.g., <For calculating the value of an algebraic expression, I can replace the variable by the specific measures in the drawing>. These theorems-in-action are shaped by *concepts-in-action,* being defined as "categories (...) that enable the subject to cut the world into distinct (…) aspects and pick up the most adequate selection of information" (ibid.), e.g., ||Variable as unique hidden number||.

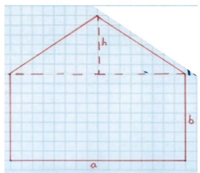
In the *second step* of analysis, we categorize the reconstructed concepts-in-action according to their subject (variable, expression, connection between expression and geometric shape, …), their degree of generality and the underlying operational or relational perspective. This allows us to identify connections that could be interpreted as sources for typical conceptual challenges (research question Q2).

RESULTS: RECONSTRUCTING CONCEPTUAL CHALLENGES

Without being able to provide wide empirical evidence from the case studies, we present short extracts of our analysis, show typical moments in the processes and discuss the connection between the reconstructed theorems- and concepts-in actions.

Episode 1 of Paula & Daniel: Degree of generality for variables and figures

Paula and Daniel (grade 9) collaborate on Task I (Fig. 1). Before Turn 62 they evaluate two given algebraic expressions as correct by relating sub-expressions to sub-areas of the figure, guided by the concept-in-action ||Relation between expression and shapes as corresponding by substructures||. For Till's expression a· b + ½ · a · h, they don't find structural correspondences and calculate instead:



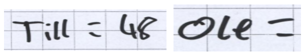
62	Paula:	So 0.5 · a · h, you need values to calculate it.
…		
66	Daniel:	Therefore, we would need this height here.
67	Paula:	Ah, just count it, don't know. That's 1 2 3 4.
…		
71	Paula:	So *a* was 8, right?
72	Daniel:	Yes
73	Paula:	8 (*writes down* ), so the half would be, that would be 4 hm *(counts the units, then counts side lengths and calculates the area)*

Paula's activities in Turn 62-73 are guided by her individual theorem-in-action: <For finding out which expression is correct, I can calculate the value of the expressions.>. Beyond it, we reconstruct her concept-in-action ||Relation between expression and shape as decidable by quantities||. Both students search for specific measures for calculating (Paula in Turn 62, Daniel in Turn 66).

Table 1. Degree of generality as a challenge on the pathway to insertion equivalence

	Specific ⟶	**General**
Conceptions for variables	hidden specific number	changing / generalized numbers
Conceptions for geometric shapes	specific drawing with fixed side length	general figure with varying side lengths
Conceptions for equivalence of expressions	equality of values = specific insertion equivalence	general insertion equivalence / general description equivalence

They solve their need by the theorem-in-action <For calculating the value of an algebraic expression, I can replace the variable by the side lengths in the drawing> (Turn 71ff), beyond which we reconstruct the concept-in-action ||Variable as hidden specific number||. Like many other students in our study, Paula und Daniel are guided by their individual focus on specific lengths. A short while later, they compare algebraic expressions by their values for a specific insertion:

93 Daniel: that was Till (*writes* 

)

94 Paula: mhm – Maybe we calculate about Ole, which result is the right one.

The theorem-in-action <For comparing two expressions, I can compare the results of the expressions> assists Paula to correctly evaluate Till’s and Ole’s expression as equivalent. However, the underlying concept-in-action ||Equivalence as equality of results|| is only partially correct, since it limits the insertion equivalence to specific numbers. Their limited degree of generality for the variables is connected to a well-known misconception for the geometric shapes: Paula and Daniel do not apply the geometric concept ||Geometric shape as general figure|| in which changeable side lengths (and as a consequence the form of the shape) are considered, but instead they apply the individual concept-in-action ||Geometric shapes as specific drawings|| (cf. Parzysz, 1988) in which side lengths are fixed to the specific drawn measures.

Paula’s and Daniel’s restriction to ||Equivalence as specific insertion equivalence|| and ||Variable as hidden specific number|| becomes an evident obstacle for developing the concept of general insertion equivalence when working with Task II where the fictitious student Till inserts several numbers for comparing the expressions in the next scene.

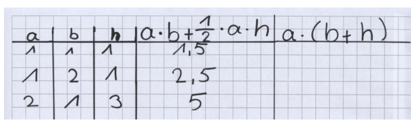
214 Paula: We filled in the right numbers and he took anyones?

…

217 Daniel: Huh? That’s not possible.

218 Teacher: Why is that impossible?

219 Daniel: You just can’t insert different numbers.



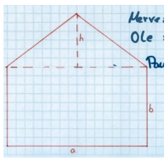
Due to their concept-in-action ||Variable as hidden specific number||, their theorem-in-action <For comparing two expressions, I can compare the results of the expressions> is limited to one insertion (the specific drawn lengths), so that they can’t get access to the general insertion equivalence.

From this snapshot and comparable episodes from other case studies, we conclude that at this point in the learning process, the individual concepts-in-action on variables and geometric shapes like those of Paula and Daniel provide a challenge for the development of conceptual algebraic understanding. Although in the later part of the design experiment, many students succeed in overcoming this challenge, we emphasize that in the first encounter, the geometric interpretation of expressions can become a source of a conceptual challenge if limited geometric understanding is activated.

Episode 2 of Jan & Niclas: Intermediate generality in operational perspectives

Jan and Niclas (grade 7) also work on Task I and start by finding an own way of calculating the area. Niclas struggles with Ole's expression a · (b + h/2).

56	Niclas:	Me, for example, I would know how to calculate the area, but the whole expression.
…	Niclas:	*(explains correctly how he would calculate the area of the drawing)*.
61	Teacher:	Mhm, just write it down anyway.
…		*(Jan wants to know, if he got right in understanding Niclas.)*
63	Niclas:	… can I just do it with units, that I count this *(he first touches the lower side and afterwards the height of the triangle)* so or just six units?
…		
66	Jan:	… there is nothing specified.
67	Niclas:	Yes
68	Jan:	There are none, so now that is, I mean, how many, let me say, that are 3 meters (*hints to side b)* that are 4 meters (*hints to side a).* It is only now that it is specified how long the sides are.
69	Teacher:	How long could they be, the sides?
70	Jan:	Different, as you can actually choose, x-variable.
71	Teacher:	mhm
72	Niclas:	Or maybe one unit as one meter, that are 16 meters *(hints to side a)* that are 9 meters *(hints to side b, gives a shrug)*, aren't they?
73	Jan:	Also possible.



Both boys operate with the individual theorem-in-action <For calculating the area of the given shape, I can insert values for the variables>, but while negotiating which value to insert, divergent concepts-in-action appear. Whereas Jan emphasizes that different values can be inserted (Turn 68, 70) and thus activates a high degree of generality, Niclas first starts with the concept-in-action ||Variable as place holder for specific numbers|| in Turn 63. Reacting on Jan's objection in Turn 72, he widens his theorem-in-action to <For calculating the area of a given shape, I can insert the side length with variable scales>. Thus, he changes his concept-in-action into ||Variable as a place holder for specific numbers but variables scales||. This concept-in-action is in line with the geometrical concept-in-action ||Geometric shape as drawing with specific side length but variable scales||.

Table 2. Operational – relational dichotomy as a challenge on the pathway to description equivalence

	Operational perspectives on variables and expressions ⟶	**Relational perspectives on variables and expressions**
Main activities	calculate	formulate, interpret, structure
Conception for algebraic expression	prompt to calculate	description for structures (e.g., pattern) for unknown / general numbers
Conceptions for variables	place holder for numbers; numbers must be inserted before dealing with expressions	specific numbers or changing / generalized numbers
Correspondence between algebraic expression and geometric shape	Relate only quantities (numbers, values <-> side length, areas)	Relate also structures (operations or subexpressions <-> substructures and parts of shape)
Conceptions for equivalence of expressions	only insertion equivalence	insertion and description equivalence

Since this concept-in-action is still restricted to geometrically *similar* drawings, we classify Niclas' concepts-in-action as having an intermediate degree of generality (locating between the columns of Table 1). With these higher degrees of generality, their further pathway to *insertion equivalence* is smoother than that of Daniel and Paula.

However, their pathway *to description equivalence* is challenged by serious difficulties in connecting the shape and the expression. The problem first appears in Turn 56, when Niclas claims not to be able to formulate his own expression. His use of variables seems to be restricted to inserting and calculating, so we reconstruct the operational concepts-in-action ||Expression as prompt to calculate|| and beyond that ||Variable as place holders||, but not ||Expression as description for structures|| (see Table 2). In contrast to Daniel and Paula who can (sometimes) activate ||Relation between expression and shapes as corresponding by substructures||, Niclas and Jan only draw connections between expressions and shapes when the expressions are written with numbers instead of variables. Later, the teacher prompts them to find sub-expressions with variables in the figure:

406 Teacher: Mhm and why did Till actually first multiply a times b and then a times h – and afterwards divide that by two? – Do you have an idea how he could have found that out?
407 Niclas: Uff – well, maybe to make it easier or something like that.
408 Jan: Well, actually he did a times h…
409 Niclas: …Because he has – he has these lengths *[hints to a and b]* or this information *[hints to the expressions]* this is what he already has, that's why you can do this … *[interrupts himself, break 8 sec.]*
410 Jan: Do you know how Ole works?
411 Niclas: Hm – no idea *[laughs]* – how you can find it out?

In Turn 409, Niclas explicitly refers to the algebraic expressions and the figure, but interrupts himself when trying to relate them to each other. The formerly used individual theorem-in-action <For connecting the shape and the expression, I can insert the side lengths> is explicitly excluded by the teacher's prompt to consider the sub-expressions with variables, but he does not find any other way to relate the shape and the expression. We draw this challenge back to a completely operational perspective of the expression, namely the concept-in-action ||Expressions as prompt to calculate|| which is directly connected to ||Variable as place holders|| (see Table 2). Turns 406-411 show how these concepts-in-action hinder the boys' capacity to relate the shape and the expression.

Although these concepts-in-action have already been located in Table 2 (for easier reading), the systematization of the observed problems and logical connections have only been conducted after comparing several cases in the last step of data analysis. Table 2 is the condensed outcome of these comparisons. It generalizes the well-known relational-operational dichotomy from the equal sign to variables and expressions. The restriction to the main activity in the operational perspective (calculating expressions) has consequences for the variable as well as for the correspondence between algebraic expression and geometric shape. For the pathway to description equivalence, operational perspectives on expressions must be complemented by relational ones that focus on own formulations of expressions, structures and interpretations. Unless the variable is considered only as place holder and the expression only as prompt to calculate, the correspondence between expressions and shapes cannot be drawn by relating substructures. In this way, the concepts-in-actions in the different lines of Table 2 are deeply connected and the transition from operational to relational perspectives is crucial.

CONCLUSION AND OUTLOOK

The empirical analysis of typical challenges showed two important dimensions in which students have to develop their initial conceptions on their pathways to a conceptual understanding for the equivalence of expressions, namely to general insertion and description equivalence: (1) the degree of generality attributed to variables and geometric shapes (Table 1; vertical axis in Fig. 2), and (2) the operational versus relational perspectives on variables, expressions and - as a consequence - the relation between expressions and geometric shapes (Table 2; horizontal axis in Fig. 2). Although the four students finally succeeded in overcoming these challenges, our extracts of their

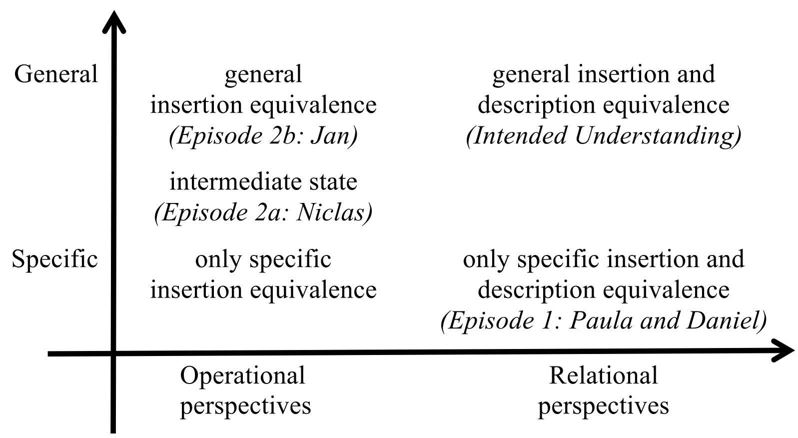


Fig. 2. Overcoming conceptual challenges in two dimensions: Overview on the cases

processes show typical moments and intermediate states of the development in these two dimensions. The first episode with Paula and Daniel shows how a specific understanding of variables and geometric shapes limits students' conceptions of equivalence of expressions. In the second episode, Jan provides a higher degree of generality, and Niclas adopts an intermediate conception on variables as specified numbers with variable scales. Jan and Niclas additionally struggle with their purely operational interpretation of expressions, variables and the connection between geometric shapes and algebraic expressions which hinders their pathway towards description equivalence, understanding expressions as equivalent when they describe the (area of the) same shape.

In the larger design research project, these findings initiated the design of additional tasks that help to overcome these challenges. For attaining higher degrees of generality, the figures are now drawn in several versions, which foster students realization that several drawings with different side lengths can belong to the same figure, and that variables signify changing lengths (cf. Prediger & Zwetzschler, 2013).

For widening students' perspectives from purely operational also to relational perspectives, we integrated tasks that focus on structural connections between geometric shapes and (first arithmetic and later algebraic) expressions by making explicit the strategies for finding substructures in expression and shapes. One example is given in Task (III) in Fig. 3 (Prediger et al., 2011). To find out which elements belong together, the students need to adopt a relational perspective. The focus on substructures is strengthened by the verbalization of strategies as a third element that serves as conceptual bridge to overcome the gap between the drawing and the expressions. The design experiments in the next cycles showed that this task encourages students to draw connections and gain hence access to the learning pathway towards understanding description equivalence.

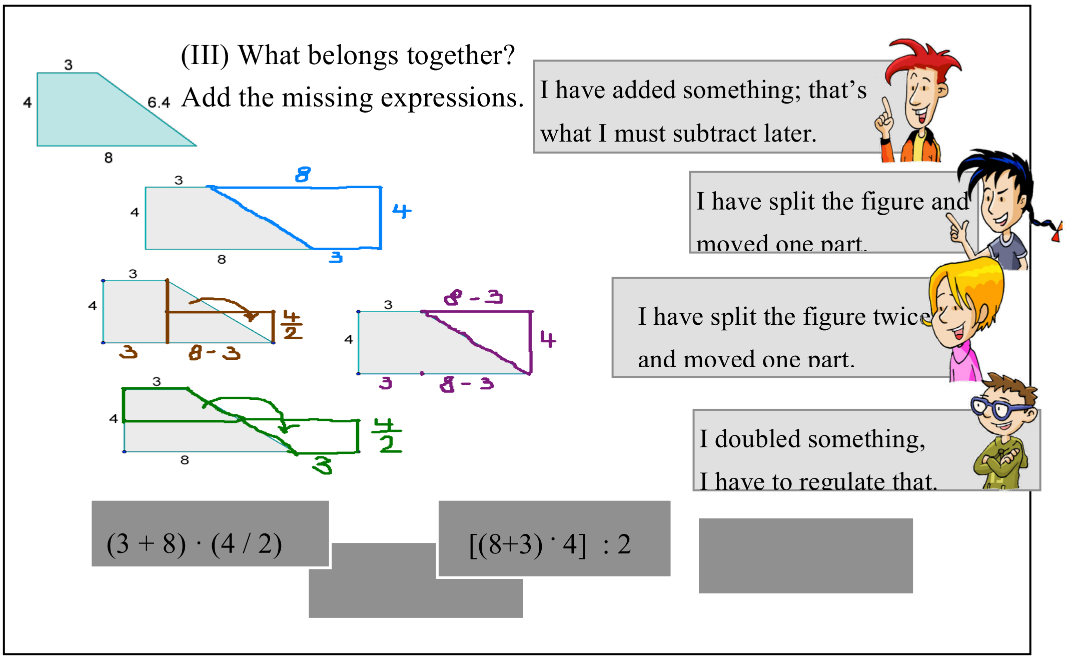


Fig. 3. Design of a task that focuses on drawing connections of substructures

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