MENTAL MATHEMATICS & ALGEBRA EQUATION SOLVING

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The reported study on mental algebraic equation solving is part of a larger research programme, aimed at better understanding the potential of mental mathematics activities with objects other than numbers. Through outlining the study details, and the activities students engaged with, I report on the variety of interpretations given to what solving an algebraic equation is. Focusing as well on the nature of students' engagement, I discuss some implications/potential for algebra teaching and learning.

CONTEXT OF THE STUDY

To highlight the relevance and importance of teaching mental calculations, Thompson (1999) raises the following points: (1) most calculations in adult life are done mentally; (2) mental work develops insights into number system/number sense; (3) mental work develops problem-solving skills; (4) mental work promotes success in later written calculations. These aspects stress the non-local character of doing mental mathematics with numbers, where the skills being developed extend to wider mathematical abilities and understandings. Indeed diverse studies show the significant impact of mental mathematics practices with numbers: on students' problem solving skills (Butlen & Peizard, 1992; Schoen & Zweng, 1986), on the development of their number sense (Boule, 2008; Murphy, 2004; Heirdsfield & Cooper, 2004), on their paper-and-pencil skills (Butlen & Peizard, *ibid.*) and their estimation strategies (Heirdsfield & Cooper, *ibid.*; Schoen & Zweng, *ibid.*). For Butlen and Peizard (*ibid.*), the practice of mental calculations can enable students to develop new ways of doing mathematics and solving arithmetic problems that the traditional paper-and-pencil context rarely affords because it is often focused on techniques that are in themselves efficient and do not create the need for doing otherwise. There is thus an overall agreement, and across contexts, that the practice of mental mathematics with numbers enriches students' learning and mathematical written work about calculations and numbers: studies e.g. conducted in US (Schoen & Zweng, *ibid.*), France (Butlen & Peizard, *ibid.*; Douady, 1994), Japan (Reys & Nohda, 1994), and UK (Murphy, *ibid.*; Thompson, 1999; Threlfall, 2002).

This being so, as Rezat (2011) explains, most if not all studies on mental mathematics focus on numbers/arithmetic. However, mathematics taught in schools involves more than numbers. This triggers interest in knowing what teaching mental mathematics with mathematical objects other than numbers might contribute to students' mathematical reasoning. In this study, issues of algebra equation solving are probed into. Thus, what learning opportunities solving algebraic equations mentally offer students? What mathematical activity do students engage in? What mathematical strategies emerge? This paper reports on the nature of strategies used by a group of 12 students, and the varied meanings developed about "solving algebraic equations".

THEORETICAL GROUNDING OF THE STUDY: AN ENACTIVIST FRAME

Recent work in mental mathematics points to the need for a better understanding and conceptualizing of how students develop mental strategies. Faced with significant varieties of students' creative solutions and with dissatisfaction about their "classification" in known and precise categories, researchers have begun to criticize the notion that students "choose" from a toolbox of predetermined strategies in order to solve problems in mental mathematics. Threlfall (2002) insists rather on the organic emergence and contingency of strategies in relation to the tasks *and* the solver (e.g. what he understands, prefers, knows, has experienced with those tasks, is confident with; see also Butlen & Peizard, 1992; Rezat, 2011). This view on emergence of strategies is also outlined in Murphy (2004), who discusses Lave's situated cognition perspective on mental strategies as flexible emergent responses, adapted and linked to a specific context and situation.

In mathematics education, the enactivist theory of cognition has been concerned with issues of emergence, adaptation and contingency of learners' mathematical activity (from the work e.g. of Maturana & Varela, 1992; Varela, Thompson & Rosch, 1991). Therefore, aspects of the theory are used to ground this study in its intention to make sense of students' strategy development and mathematical activity. In particular, Varela's (Varela et al., 1991) distinction between *problem posing* and *problem solving* offers a preliminary answer to questions about emergence and the characterization of strategies generated for solving tasks.

For Varela, problem-solving implies that problems are already in the world, lying "out there" waiting to be solved, independent of us as knowers. In contrast, Varela explains that we specify, we pose, the problems that we encounter through the meanings we make of the world in which we live, which leads us to recognize things in specific ways. We do not "choose" or "take" problems as if they were lying "out there," objective and independent of our actions: we bring them forth. The problems that we encounter and the questions we ask are thus as much a part of us as they are a part of our environment: they emerge from our interaction with it, as we interpret events as issues to address, as problems to solve. Thus, we are not acting on preexisting situations: our interaction with the environment creates the possible situations for us to act upon. The problems that we solve, then, are implicitly relevant for us because we allow these to be problems for us.

Hence it is claimed that reactions to a task do not reside inside either the solver or the task: they emerge from the solver's *interaction* with the task, through posing the task. If one adheres to this perspective, one cannot assume, as René de Cotret (1999) explains, that instructional properties are present in the (mental mathematics) tasks offered and that these will determine solvers' reactions. Strategies emerge in the interaction of solver and task, influenced by the task but determined by the solver's experiences and understandings: in his solving habits for similar or different tasks, in his successes in mathematics with specific approaches, in his understanding of the tasks, etc. In this perspective, the solver does not choose from a group of predetermined strategies to solve the task, but engages with the problem in a specific way and develops a strategy tailored to the task he poses.

Thus, students *transform* the mathematical tasks for themselves, making them their own, which is often different from the designer's intentions (René de Cotret, 1999); Heirdsfield and Cooper (2004) and Rezat (2011) have indeed shown the occasional futility in mental mathematics of varying the type of problem or its didactical variables to encourage students to use specific strategies. When solving tasks, students generate a strategy tailored to the problem (they) posed, as acts of posing and solving are not predetermined but generated in interaction with the task:

As a result of this interaction between noticing and knowledge each solution 'method' is in a sense unique to that case, and is invented in the context of the particular calculation – although clearly influenced by experience. It is not learned as a general approach and then applied to particular cases. […] The 'strategy' (in the holistic sense of the entire solution path) is not decided, it emerges. (Threlfall, 2002, p. 42)

Students are then seen to generate their strategies in order to solve their tasks. These are adapted responses, locally tailored to the tasks, emerging in interaction with them.

THE STUDY – DEFINING MENTAL MATHEMATICS

Because most work on mental mathematics is on numbers (often referred to as *mental arithmetic* or *mental calculations*) and defined accordingly, no formal comprehensive definition of mental mathematics appears in the literature. Based on the work on mental calculations, one tentative definition is: Mental mathematics is the solving of mathematical tasks without paper and pencil or other computational/material aids. This definition helps understanding the "constraints" to which the students are subjected to, the major issue being that students do not have access to any material aid, be it paper-and-pencil or other, to depend on for solving the problems offered to them. This study focuses on (any of) the strategies produced in this context.

METHODOLOGY, DATA COLLECTION AND ANALYSIS

One intention of the overarching research programme is to study the nature of the mathematical activity students engage in through working on mental mathematics. This is probed through (multiple) case studies, taking place in educative contexts designed for the study (classroom settings/activities). The reported study is one of those case studies, in a university mathematics education course. This site was aimed for because these students are not novice solvers in algebra, enabling a focus on their solving of algebra tasks (and less on their difficulties with algebra itself).

Classroom activities were designed to offer algebraic equations for students to solve mentally. A variety of algebraic equations of the form $\mathrm{A}x+\mathrm{B}=\mathrm{C}$, $\mathrm{A}x+\mathrm{B}=\mathrm{C}x+\mathrm{D}$, $\mathrm{A}x/\mathrm{B}=\mathrm{C}/\mathrm{D}$, $\mathrm{A}x^2+\mathrm{B}x+\mathrm{C}=0$ and their variants were presented. The classroom organization took the following structure: (1) an equation is offered orally or in writing on a transparency to the group; (2) students solve the equation mentally (without paper-and-pencil or material aids to solve or leave traces); (3) at the signal they write their answer on a piece of paper; (4) answers and strategies are orally shared, noted on the transparency. The data collected comes from the strategies orally explained (and noted on transparencies), as well as notes taken after the session.

The data was first looked at, analyzed, in relation to the nature of the strategies generated by students for solving the tasks. Because this analysis is dependent on the type of mathematical objects worked with through the classroom activities, available theoretical concepts found in the literature to guide and enhance the data analysis were used. In algebra, unwinding/undo procedures (Nathan & Koedinger, 2000) or transformation of equations (Arcavi, 1994), to name but two, are examples of relevant dimensions used for the data analysis. This analysis rapidly led to considerations of the meanings given to solving an algebraic equation, the focus of this paper (other analyses regarding the strategy dimensions are to appear in another paper). Following Douady (1994), the goal of this paper is not to report on all learning that took place for students, nor to discuss the long-term outcomes for students in other contexts, but mainly to understand the meaning and functionality of the tools used (i.e. strategies for mentally solving algebraic equations) and explore their potential. The focus in this paper is thus placed on the problem posing aspects, that is, the nature of the mathematical strategies engaged in and its repercussions on the meanings afforded to what solving an algebraic equation is (see Bednarz, 2001).

FINDINGS – MEANINGS FOR ALGEBRAIC EQUATION SOLVING

Through solving the various tasks offered to them, students gave, implicitly, different meanings to what solving an algebraic equation represents. Those meanings are mathematically rich and contribute to deepen understandings of what solving an algebraic equation is. I outline below these various meanings.

***Meaning 1*: finding the value(s) that satisfy, make true, the equality**

Underneath this meaning is the notion of a conditional equality, where it is not only the idea of finding the answers/values that make the equation true, but also the fact that the equality *can* be true or untrue.

When students were given $5x+6+4x+3=-1+9x$ to solve, some rapidly asserted that there was no solution, because one can rapidly see $9x$ on both sides of the equation as well as the fact that the remaining numbers on each sides do not equate. It thus leads to the conclusion that there was no number that could satisfy the given equation, since no x, whatever it could be, could succeed in making different numbers equals. This strategy is related to what is often termed "global reading" of the equation (Bednarz & Janvier, 1992), that requires consideration of the equation as a whole prior to entering in algebraic manipulations, or what Arcavi (1994) calls *a priori* inspection of symbols, which is a sensitivity to analyze algebraic expressions before making a decision about their solution. (Arcavi gives the example of

$(2x+3)/(4x+6)=2$, which has no solution because whatever the value of x, the numerator is worth half the denominator, making futile undergoing additional steps.)

Another strategy students engaged in was one of "solving followed by validation". When having to solve $x^2–4=5$ one student rapidly transformed it into $x^2=9$, obtaining 3 as an answer. However, because he is in a mental mathematics context and is aware that his answers in this context are often rapidly enunciated and can lack precision, he decides to verify his answer. By mentally verifying if $(3)^2=9$, he realizes that $(–3)^2$ also gives 9 and then readjusts his solution. This manner of solving the equation gets close to the idea not only of finding one value that makes the equation true, but also of finding *all* values that make it true.

***Meaning 2*: deconstructing the operations applied to an unknown number**

This meaning requires reading the equation as a series of operations applied to a number (here x) and attempting to undo these operations to find that number.

When having to solve equations like $x^2–4=5$, students would say: "My number was squared and then 4 was taken away, thus I need to add 4 and take the square root". Or, for $4x+2=10$, "What is my number which after having multiplied by 4 and added 2 to it gives me 10?" These are similar to inverse methods of solving found in Filloy and Rojano (1989) and Nathan and Koedinger's (2000) "unwinding", where operations are arithmetically "undone" to arrive at a value for x. As Filloy and Rojano explain, when using this method "it is not necessary to operate on or with the unknown" (p. 20), as it becomes a series of arithmetical operations performed on numbers. In this particular case, solving the algebraic equation is focused on finding a way to arrive at isolating x, in an arithmetic context.

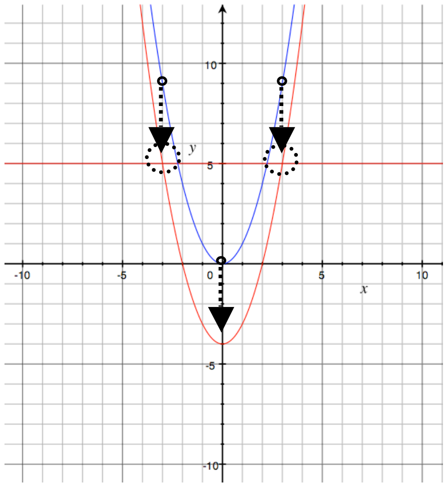
Meaning 3*: operating identically on both sides to find *x

This meaning focuses on the idea that is often called "the balance" principle, where one operates identically on both sides of the equation to maintain the equality and obtain "x=something". For example, when solving $2x+3=5$, students would subtract 3 on each side and then divide by 2.

***Meaning 4*: finding points of intersection of a system of equations**

This is about seeing each sides of the equality as representing two functions, and thus attempting to solve them as a system of equations to find intersecting points, if any.

For example, when solving $x^2–4=5$, some students attempted to depict the equation as the comparison of two equations in a system of equations ($y=x^2–4$ and $y=5$) and finding the intersecting point of those two equations in the graph. To do so, one student represented the line $y=5$ in the graph and then also positioned $y=x^2–4$. The latter was referred to the quadratic function $y=x^2$, which crosses $y=5$ at $x = \sqrt{5}$. In the case of $y=x^2–4$, the function is translated of 4 downwards in the graph, and then the 5 of the line $y=5$ becomes a 9 in terms of distances. Hence, how does one obtain an image of 9 with the function $y=x^2$? With an $x=\pm3$, where the function $y=x^2–4$ cuts the line $y=5$. The following graph offers an illustration of what the student did, mentally.



Solving an algebraic equation in this case is not about finding the values that make the equation true, but about finding the x that satisfies both equations for the same y, about finding the x coordinate that, for the same y, is part of each function

***Meaning 5*: finding the values that nullifies the equation**

This meaning focuses on the equal sign as giving an answer (see e.g. Davis, 1975), but where operations are conducted so that all the "information" ends up being on one side of the equation to obtain 0 on the other side. The intention then becomes to find the value of x that nullifies that equation, that is, that makes it equal to 0.

One example of a strategy engaged in was again about seeing the equation in a function view as in meaning 4, but here for finding the values of x that give a null y-value, or what is commonly called finding the zeros of the function where the function intersects the x-axis at y=0. For $x^2–4=5$, transformed in $x^2–9=0$, the student aimed mentally at solving $(x+3)(x–3)=0$, leading to ±3. The quest was mainly finding the values that nullify the function $y=x^2–9$, which gave point(s) for which the image of the function was zero. Another way of doing it, less in a function-orientation, is to use "binomial expansion" (what is called in French *identités remarquables*) for seeing that for the product to be null it requires that one of the two factors be null. This said, one needs to use neither a function nor binomial expansion to find what nullifies the equation. For example, if $x+4=3$ is transformed in $x+1=0$, one finds that –1 is what makes the left side of the equation equal to 0.

***Meaning 6*: finding the missing value in a proportion**

This meaning was engaged with for equation written in fractional form (e.g. Ax/B=C or Ax/B=C/D). In these cases, the equation was conceived as a proportion, where the ratio between numerators and denominators was seen as the same or consistent. In this case, the equality is not seen as conditional but is taken for granted, to be true, leading at conserving the ratio between numerator and denominator in the proportion.

For example, for $\frac{6}{x}=\frac{3}{5}$, reversed to $\frac{x}{6}=\frac{5}{3}$, students solved by saying "If my number is 6 times bigger than x/6, then it is 6 times bigger than 5/3". Another way offered was to analyze the ratio between each numerators and apply it to denominators which had, in order to maintain the equality, to be of the same ratio: "If 6 is the double of 3, then x is the double of 5 which is worth 10".

***Meaning 7*: finding equivalent equations**

This meaning for solving the equation is oriented toward obtaining other *equivalent* equations to the first one offered, in order to advance toward an equation of the form "x=something". This is related to Arcavi's (1994) notion of knowing that through transforming an algebraic expression to an equivalent one, it becomes possible to "read" information that was not visible in the original expression. Through these transformations, the intention is not directly to isolate x, but to find other equations, easier ones to read or make sense of, in order to find the value of x.

An example of such was done when solving $\frac{2}{5}x=\frac{1}{2}$, where some students doubled the equation, obtaining $\frac{4}{5}x=1$, which was simpler to read and then multiplied by 5/4 to arrive at x=5/4. This is an avenue also reminiscent of arithmetical divisions, where equivalences are established: e.g. 5.08÷2.54 is equivalent to 508÷254, because 254 divides into 508 the same number of times as 2.54 into 5.08.

Similarities and differences in meanings attributed

Albeit treated separately, these varied meanings are not all different and some share attributes. Therefore, in addition to the variety of meanings, significant links can be traced between those, links that can deepen understandings about algebraic equation solving. For example, meanings 2 and 3 share an explicit orientation toward isolating x, where others do not have this salient preoccupation and focus on other aspects (satisfying the equality, finding points of intersection, etc.). Meanings 4 and 5 share a function orientation in their way of treating the equation, emphasizing each part of the equation as representing an image (or simply the value of the function).

Many meanings also focus implicitly on conditional orientations, be it concerning the satisfaction of the equality or simply the possibility of finding a value for x. For example, in meaning 2 and 3, it is possible that no value of x is found and the same can be said for meaning 4, where it is possible that there be no point of intersection of the two equations or for meaning 5, where possibly no value of x could nullify the left side of the equation (e.g. $x^2+\sqrt{2}=0$). Without being explicit about it, these orientations represent a quest for finding a possible value, a quest that can be unsuccessful. This contrasts heavily with meaning 6, because treating the equation as a ratio assumes or implies that a value of x exists. Meanings 1 and 6 however do share something in common, which is related to an examination of relations between the algebraic unknown and the numbers in order to deduce the value of the algebraic unknown. Both do not opt for a sequence of steps to undertake, but mainly for working with the equation as a whole (in global reading for meaning 1, in ratios for meaning 6). Meanings 3 and 7 share the fact that operations are conducted on the equation as a whole, be it through affecting both sides in the same way to keep the "balance" intact or to obtain new equivalent equations.

Finally, meanings 1, 2, 5 and 6 share the fact that they explicitly look for a number, where the algebraic unknown is conceived as an unknown number that needs to be found; a significant issue to understand when solving algebraic equations (Bednarz & Janvier, 1992; Davis, 1975). Hence, be it through looking at which number could satisfy the equation (meaning 1), which number could nullify a part of it (meaning 5), which number satisfies the proportion (meaning 6) or which is the number for which operations were conducted (meaning 2), all of them focus on *x* as being a number.

DISCUSSION OF FINDINGS

On the emergence of mental mathematics strategies

The variety of meanings brought forth through students' solving illustrates well how the various "posing" of the problems led to various strategies for solving and thus to various meanings attributed to algebraic equation solving. Each equation provoked numerous strategies for solving it, leading to numerous meanings attributed to algebraic equation solving. Thus, the same equation made emerge a variety of posings, of strategies, of meanings. This supports the view that strategies for solving emerge in the interaction of solver and task, where the solver plays an important role as he poses the tasks, and where the nature of the task plays a role as well, with a strategy tailored to it (see e.g. the impact of fractional or second degree forms on the nature of the strategies). Strategies emerge contingently where, as Davis (1995) explains, they are inseparable from the solver and the task, emerging from their interaction. Thus, building on Simmt's (2000) work, the tasks given were not tasks but mainly prompts for solvers to create tasks with: prompts were offered to students, not tasks. Tasks became tasks when students engaged with them. Students *made* the equations ones about system of equations, about functions, about ratio, etc., allowing a variety of meanings for algebraic equation solving to emerge along the way.

On the potential of mental mathematics for algebra

This variety of meanings, emerging with/in students' posing, has enormous potential for algebra teaching and learning. These meanings are significant, because they offer different entry paths into the tasks of solving algebraic equations and do not restrict a single view of how this can be done. Numerous authors have outlined difficulties experienced by solvers (from school to university) in solving algebraic equations (see Bednarz, 2001; Filloy & Rojano, 1989; Nathan & Koedinger, 2000). The emergence of this variety of meanings offers significant reinvestment opportunities for pushing further the understanding of algebraic equation solving. This is related to Butlen and Peizard's (1992) assertion that the practice of mental calculations can enable students to develop new ways of doing mathematics and solving arithmetic problems that the traditional paper-and-pencil context rarely affords because it is focused on techniques and algorithms that are in themselves efficient and do not create the need for stepping outside of them. This seems also to be the case here for mental algebraic solving. Issues of conditional equations, of deconstructing an equation regarding operations done on a number, of maintaining the balance, of finding equivalent equations, of seeing an equation as a system of equations, and so forth, offered varied ways of conceiving an equation and of solving it. It opened various paths of understanding.

Without making paper-and-pencil a straw-men for criticism, the mental mathematics context can be seen to have provoked some strategies and meanings different than ones used in the usual written context for solving equations in algebra. An example is the transforming of equations into equivalent ones (e.g. $\frac{2}{5}x=\frac{1}{2}$ to $\frac{4}{5}x=1$). This reasoning is at the basis of solving equations in writing. But, it was here quite different than the usual transformations applied to equations, since the equation was not transformed in order to isolate x, but mainly to obtain other equations that were easier to read. In fact, even if many strategies were reminiscent of paper-and-pencil work, the major difference is that there was no paper-and-pencil work, leading to a dialogue taking place between the student and the task, *while* solving. Because students could not leave written traces or transform the equations in writing, and thus could not interact with what was obtained after each written step through manipulating the equation, the monitoring of the solution was done in real-time, through personal dialogue, through a story told in which the solver engaged through telling the story, to keep track of the operations being conducted and their adequacy. In short, students had to invent (to pose) their stories about the problem, to interpret the equation in their own terms, in order to find a way to solve it. These "steps", oriented by the task at hand, oriented in return the next steps. Solving was done through the action of solving, and not through applying a known procedure, making it quite a different solving experience.

This mental algebra context seems to have offered a different context for solving, one that led to the development of various ways of solving and making sense of algebraic equations. It opened spaces of exploration that can be taken advantage of in teaching, in order for example to unearth the various meanings given to solving algebraic equations or similarities and differences between those. The variety of meanings that emerged also shows how the mental mathematics context offered occasions for thinking differently about algebraic equation solving. The opening of this varied solving space, not restricted to a one-size-fits-all way of solving, can have significant impact on students' understandings of and attitudes toward algebraic equation solving. Obviously, these meanings emerged in this specific context, with these students, and there is no guarantee that this would be so in another context. But this is not the point. The point is to generate deeper understandings of the sort of learning opportunities that solving algebraic equations mentally can offer, and to know more about the nature of the strategies engaged with. Thus, in the context of the study, with students studying to become future teachers, these openings to varied ways of solving and of conceiving algebraic equations cannot be underestimated. More research is still needed, but already this emerging variety of meanings shows important promise of mental mathematics for enriching algebraic experiences.

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