LEVELS OF THINKING AND CRITICAL ASPECTS

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In this article, the intention is to make a contribution to the existing research on algebra learning by presenting the van Hiele levels of reasoning and adapt them to algebra through use of the variation theory. The collected data consists of: students' tests, examinations of students' mathematical work, the teachers' lesson plans and reports of the lessons' instructions. The results indicate that there are different levels of thinking which have their own network of relations. The transition from one level to the next can be realised by analysing the real critical aspects in students' learning and opening up for dimensions of variation in those aspects. The findings suggest valuable implementation to develop effective ways of experiencing the object of learning and expand students' algebraic abilities.

Keywords: algebra, critical aspects, dimensions of variation, variation

INTRODUCTION

In recent years, more and more researchers (e.g., Marton & Tsui, 2004; Olteanu, 2007; Olteanu & Olteanu, 2010, 2012) have become interested in studying the relation between teaching and learning. They also attempted to enhance their mathematics teaching to be more meaningful and powerful in various ways. Several theories have been used for this purpose. One of these theories is the variation theory (Marton & Booth, 1997; Marton & Tsui, 2004) which can be used to study the relation between the learner and what is learned. Another theory, developed by van Hiele (1986), aims at providing a basis for understanding the movement between levels of thinking and the role of the teacher in assisting with such progression. This theory is usually used to improve teaching of geometry. Our first attempts in this article are to study the improvement of teaching and learning of algebra.

According to van Hiele, learning occurs when students experience a "crisis of thinking" (van Hiele, 1986, p. 43). Learning, according to the variation theory, is defined as a change in the way a person experiences a particular phenomenon and it is associated with a change in discernment in that person's structure of awareness. Olteanu (2012) specify that, there is a change in the critical aspect(s) of the phenomenon that the learner simultaneously focuses on after that learning has taken place. The learner is able to discern critical aspects that he or she could not discern before. From a variation theory perspective, learning is seen as a function of how the learner's attention is selectively drawn to critical aspects of the object of learning. Olteanu (2012) points out that a critical aspect is the capability to discern aspects presented, for example in algebraic structures, by experiencing them. She found that, to experience a rational expression is to experience both its meaning, its structure (composition) and how these two mutually constitute each other. So neither structure nor meaning can be said to precede or succeed the other. van Hiele (1986) describes the concepts of structure as a "network of relations" in which commonalities are recognised across all types of events and perceptions.

The fundamental purpose of this paper is to test the ability of the van Hiele theory to describe levels of thinking using critical aspects in algebra. The research questions in this article are: (1) Is there a relationship between van Hiele levels and critical aspects? (2) To what extent are van Hiele levels related to critical aspects?; (3) Which dimensions of variation open up in the van Hiele stages of learning, and what aspects of the object of learning do students distinguish?

VAN HIELE LEVELS AND CRITICAL ASPECTS

The notion of critical aspects is a key concept in variation theory (Marton & Booth, 1997; Marton et al., 2004). The central idea in variation theory is that to discern certain aspects of an object of learning, a person needs to experience variation corresponding to those aspects. These are called critical aspects or critical features as long as the student has not discerned them (Olteanu & Holmqvist, 2009). A critical aspect is not the same as a difficulty, but one can identify critical aspects among the difficulties that students exhibit in mathematics (Olteanu, 2012, Olteanu & Olteanu, 2012).

In an effort to improve and understand the students' learning, van Hiele (van Hiele, 1986) proposed five hierarchical levels (0-4) that describe growth in student thinking in geometry. Hoffer (1981) describes the levels in the following way:

> **Level 0** (Visualization). The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.
>
> **Level 1** (Analysis). The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established.
>
> **Level 2** (Informal deduction). The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.
>
> **Level 3** (Deduction). The student reasons formally within the context of a mathematical system, complete with undefined terms, axioms, an underlying logical system, definitions, and theorems.
>
> **Level 4** (Rigor). The student can compare systems based on different axioms and can study various geometries in the absence of concrete models.

Olteanu and Olteanu (2010) introduced the concept of real and potential critical aspects. Potential critical aspects (PCA) or intended critical aspects are what teachers suppose to be critical aspects of students' learning, while real critical aspects (RCA) or lived critical aspects are what students' exhibit as critical aspects in their learning, which is what students do not discern. Olteanu and Olteanu (2010, 2012) found six general categories that can be used to analyse the relation between the intended and lived object of learning. Olteanu (2012) describes these general categories in the following way:

> **Category A** (The whole). The student can discern the relation between the numerator and denominator (the algebraic expression).
>
> **Category B** (The parts). The student can discern the composition of numerator and denominator.
>
> **Category C** (The relations between the parts). The student can discern the operation between term in the numerator and/or in the denominator and the relation between the numerator and denominator.
>
> **Category D** (The transformation between the parts). The student can discern to factorise the numerator and/or in denominator.
>
> **Category E** (The relation parts-whole). The student can discern the cancelling.
>
> **Category F** (The relation between different wholes). The student can discern the equivalent relation between two algebraic expressions.

One example of how these categories work with the specific content that refers to simplifying rational expressions is presented in Table 1.

The intended object of learning refers to the part of the content that students should learn and which is supposed to be treated in the classroom. The students' initial level of capability to the appropriate object of learning as well as the way in which students understand the object of learning is the lived object of learning. The object of learning that is the focus in a teaching situation. An object of learning has two constituent parts: the direct and indirect objects of learning. The first part is defined in terms of content, and the latter refers to the specific capability that students are expected to develop. What is possible for students to experience within a learning environment is called the enacted object of learning.

As mentioned earlier, van Hiele (1986) describes the concepts of structure as a "network of relations" and insight as a recognition of structure. He distinguishes between rigid and feeble structures (van Hiele, 1986, p.19-23). For example, if a student sees the expression, $\frac{4x^2-12x+9}{4x^2-9}$, it is more likely for the student to attempt to factor the numerator and the dominator and simplify the expression. This may occur if the student has at least moderate algebraic ability. The recognition of the requirement to factor the numerator and dominator may be thought of as a relatively rigid structure. Recognising such an expression provides an opportunity for factorisation. Using words such as "quadratic" and "difference of two squares" and being able to recognize their structure however, is likely to be experienced as a relatively feeble structure by most students.

To discern certain aspects of the object of learning, a person needs to experience variation corresponding to those aspects (Marton et al., 2004) in the enacted object of learning, that is what appears in the classroom and refers to what is possible for students to experience within a learning environment, i.e. was made attainable through actual patterns of variation and invariance. Marton, Runesson and Tsui (2004) have defined the patterns of variations which can facilitate students' discernment of critical features or aspects of the object of learning: (1) contrast (C) means that to discern a quality X, a mutually exclusive quality non X needs to be experienced simultaneously (e. g., $\frac{1}{x}$ and x); (2) the meaning of separation (S) is that in order to discern a dimension of variation that can take on different values, the other dimensions of variation need to be kept invariant or varying at a different rate (e.g., $\frac{2x+12}{2x} = \frac{2(x+6)}{2x}$); (3) generalisation (G) means that to discern a certain value, X_1, in one of the dimensions of variation X from other values in other dimensions of the variation, X_1 needs to remain invariant while the other dimensions vary (e.g., specify that only factors and not terms, can be cancelled); (4) fusion (F) is to experience the simultaneity of two dimensions of variation (e.g., simplify a rational expression in which the numerator and the denominator are polynomials of grad two). Olteanu and Olteanu (2012) have found a new dimension of variations named similarity (SI) and it is defined as the property of two or more expressions to adapt the same meaning.

The theory of van Hiele offers five stages of learning as a method for organizing instruction, content and materials used. These stages are: (1) information - students get acquainted with the working domain (e.g. the teacher and students engage in conversation and activity about the objects of learning and the rational expressions are separated from other algebraic expressions by seeing the numerator and denominator); (2) guided orientation - students are guided by tasks (given by the teacher, or made by themselves) with different relations of the network to be formed (e.g., .g. the students explore the numerator and dominator and the relations between them through materials that the teacher has carefully sequenced); (3) explicitation - students become conscious of the relations, they try to express them in words, they learn the technical language of the subject matter (e.g., students express and exchange their emerging views about the structures characterizing rational expressions, and discuss with each other and the teacher to the relation between the numerator and denominator or the equivalent relation between two rational expressions); (4) free orientation - students learn by general tasks to find their own way in the network of relations (e.g., simplify fractions with polynomials in the numerator and denominator by factorising both); (5) integration, they build an overview of all they have learned of the subject, of the newly formed network of relations now at their disposal (e.g., the students identify different sets of properties that characterise a class of rational expressions and identifies minimum sets of properties that can characterise a rational expression).

METHODOLOGY

In the present paper we discuss and report the results of students' learning and the teaching of Mathematics C. The presentation is based on data collected, during a 3-year period, in a development project. The presentation of the project is not described in this paper because of limited number of pages, but a detail presentation can be found (e.g., Olteanu & Olteanu, 2010, 2012). Two teachers (here called Thomas and Patrik) and 65 students (23 in phase I, 18 in phase II, and 24 in phase III) were selected from the Natural Science Programme and participated in the project. The analysis is grounded in 30 exercises and 12 written reports.

The data was collected in 11 steps which recurred in three phases. The teachers examined the course module and curriculum to identify the intended object of learning (Step 1). The teachers identified the object of learning which in this article, is to simplify a rational expression (Step 2). The project continued by explaining various concepts used in the variation theory to the teachers and putting those concepts into practice (Step 3-4). Then, the teachers worked to identify potential critical aspects in students' learning (Step 5). Subsequently, tests and interviews were conducted with students to identify the real critical aspects of their learning (Step 6-7). Based on the identified real critical aspects and the difference between potential and critical aspects, the key concept of the theory of variation was explained again (Step 8). The teachers implemented six lessons (Step 9). After each lesson, the teachers wrote a detailed report using the following template: (I) General information: school, class/group, teacher, moment, object of learning, type of lesson; (II) General purpose; (III) Specific purpose: content, emotional view, psychomotor view; (IV) Prerequisites: technical aids, materials; (V) Lesson implementation according to teaching method (with focus on the open dimensions of variation) and activities with students (Step 10). The students took different tests after the implementation of the lessons (Step 11).

RESULTS

Initial analysis entailed coding student responses for types of discerned aspects and teacher reports for types of focused aspects. In the first phase of the project the teachers worked together to identify the potential critical aspects in students' learning and to create the intended object of learning on the basis of these identified aspects. Their work was documented in written reports based on the following questions: What aspects are discerned by the students when simplifying rational expressions? What dimensions of variation can be opened up in the aspects that are not discerned by the students?

At the beginning of the project (phase I), the teachers did to a large extent suppose that students did not discern rational expressions as a whole (A), the relation parts-whole (E) and the relation between different wholes (F). However, they did not consider that students need to better understand the constituting parts (B), the relation between those parts (C) and how to relate the parts to each other in a different way

(D) (Table 1). The teachers did for example assume that students can discern the difference between terms and factors, and that only common numerical or algebraically factors would be cancelled in the simplification of a rational expression. Consequently, the teachers' intent was to focus on the aspects in categories A, E and F and less or not at all on those in categories B, C and D. In addition, it was only from time to time or rarely that they focused on opening up dimensions of variation in these aspects.

In the first phase of the project, none of the students discerned the aspects that teachers expected them to do. In the text above we have seen the difference between two levels: on the lowest level, the visual level (category A), rational expressions are recognised by discerning the fraction line between the numerator and denominator. At this level, students identify a rational expression by its appearance as a whole in a simple expression, in different expressions or in more complex expressions. The students' name or label rational expressions using standard or non-standard names and refer to numerator and denominator. The students view the rational expressions as total entities rather than as having components or attributes.

On the higher level (the analyse level) a rational expression is recognised by its properties (categories B and C): being able to factorize the polynomial expressions if they for example appear as "quadratic" and "difference of two squares". An analysis of rational expressions begins, for example, through observation and experimentation. By doing this, students begin to discern the characteristics of rational expressions. Rational expressions are recognized as having parts and they are recognized by their parts.

At the next level, the informal deduction level, (category D) the students identify different sets of properties that characterise a class of rational expressions and identifies minimum sets of properties that can characterise a rational expression. Furthermore, the students explain that two different rational expressions have the same structure and having drawn a conclusion, justify the conclusion using logical relations. The students can establish the interrelationships of properties of a rational expression and recognize classes of rational expressions, but students do not see how the logical order could be altered.

In the level of deduction (category E) the interrelationship and role of undefined terms, definition, and condition of existents is seen by the student. A student at this level can discern, not just memorize, the possibility of simplification of rational expressions in more than one way; the interaction of necessary and sufficient conditions is understood; distinctions between a statement and its converse can be made.

In the last level, rigor, (category F) rational expressions are seen in the abstract, and the students can work with rational expressions in a variety of mathematical domains, such as function, derivative, geometry and so on.

Table 1: Categorisation of aspects and examples of non discerned aspects (Olteanu, 2012).

Categories	Rational expression	Explanation	Examples of non discerned aspects
the whole (A)	$\frac{4x^2-12x+9}{4x^2-9}$	to discern/focus on the relation between the numerator and denominator (the algebraic expression)	$\frac{4x^2-12x+9}{4x^2-9} \Rightarrow -12x-1$
the parts (B)	$4x^2 - 12x + 9$ and $4x^2 - 9$	to discern/focus on the composition of numerator and denominator	$\frac{a^2-b^2}{5(a-b)} \Rightarrow \frac{a-b}{5}$
the relations between the parts (C)	$\frac{4x^2-12x+9}{4x^2-9}$	to discern/focus on the operation between term in the numerator and/or in the denominator and the relation between the numerator and denominator	$\frac{a^2-b^2}{5(a-b)} \Rightarrow$ $\Rightarrow \frac{a\cdot a-b\cdot b}{a\cdot a\cdot a\cdot a\cdot a-b\cdot b\cdot b\cdot b\cdot b} =$ $\Rightarrow \frac{1-1}{a\cdot a\cdot a-b\cdot b\cdot b}$
the transformations between parts (D)	$\frac{(2x-3)^2}{(2x+3)(2x-3)}$	to discern/focus on to factorise the numerator and/or in denominator	$\frac{4x^2-12x+9}{4x^2-9} \Rightarrow$ $\Rightarrow \frac{(2x+3)(2x+3)-12x}{(2x+3)(2x-3)} \Rightarrow$ $\Rightarrow \frac{(2x+3)-12x}{2x-3}$
the relation between parts and whole (E)	$= \frac{(2x-3)^{\cancel{2}}}{(2x+3)\cancel{(2x-3)}}$	to discern/focus on cancelling	$\frac{7x+x}{7x+3x} \Rightarrow \frac{1}{3}$
the relation between different wholes (F)	$\frac{4x^2-12x+9}{4x^2-9} =$ $= \frac{(2x-3)^2}{(2x-3)(2x+3)} =$ $\frac{2x-3}{2x+3}$	to discern/focus on the equivalent relation between two algebraic expressions	$\frac{30x+x^2}{x} \Rightarrow$ $\Rightarrow \frac{30x^3}{x} \Rightarrow 30x^2$

In phase II and III the students improved their ability to discern different aspects of a rational expression. An explanation for this phenomenon is that in phase II, the teachers focused on opening up dimensions of variation in the identified real critical aspects. In six consecutive lessons, Thomas focused on several aspects and opened up dimensions of variation by separation (S), contrast (C), generalisation (G) and fusion

(F) using several tasks in which the numerator varies and the denominator is kept invariant or vice versa. Some examples of the dimensions of variation opened up are:

- the difference between a fraction with unitary numerator and a nonfraction (e. g. $\frac{1}{x}$ and x); (C)
- the difference between factorising a polynomial and solving an equation (e. q. $2x + 12$ and $2x + 12 = 0$); (C)
- the difference between terms and factors (e. g. $2 + x$ and $2x$); (C)
- specify multiple times that only factors and not terms, can be cancelled; (S, G)
- identifying the common factor in the numerator and denominator; (S, G, SI)
- the common factors can be simplified by any common numerical or variable factors (e.g. $\frac{2x+12}{2x} = \frac{2(x+6)}{2x}$); (C, S, G, F, SI)
- the use of parentheses around the numerator and denominator to highlight the whole; (C, S, G, F, SI)
- simplify fractions with polynomials in the numerator and denominator by factorising both and renaming them using the lowest terms (e. g. $\frac{2x+12}{2x+14} = \frac{2(x+6)}{2(x+7)}$); (C, S, G, F, SI)
- identifying and factorising the difference of two perfect squares (S, G, F).

In addition, Thomas kept the following questions in the communication that occurred in the classroom invariant: What does factorising look like for a polynomial expression?; How do we know when we are finished factorising?; What is the process we use to cancel?; What does cancelling look like?; When do we know we are finished cancelling? All these questions have the same meaning, thus Thomas opened up a dimension of variation by similarity.

During the six lessons, the teacher and students participated in conversation and activity concerning the objects of learning. With focus to the critical aspects of the object of learning observations were made, questions were raised, and level-specific vocabulary was introduced, namely an introduction of the object of learning was made. In this way, students get acquainted with the working domain. The students are guided by tasks (given by the teacher, or made by themselves) focusing on forming different relations of the network in the object of learning. They have the opportunity to explore the topic of study through materials that the teacher has carefully sequenced. Building on their previous experiences, students express and exchange their emerging views about the structures that have been observed. They become conscious of the relations, they try to express them in words and they learn the technical language of the subject matter. The student encounters more complex tasks-

tasks with many steps, tasks where the students may discern different ways of solving said tasks. By orienting themselves in the field of investigation, many relations between the objects of learning become explicit to the students. The students learn to find their own way in the network of relations through general tasks. Students build an overview of all they have learned of the object of learning, and connect the newly formed network of relations to other, already existing relations. The teachers focus to the real critical aspects in students' learning and to open up for dimensions of variation in those aspects contribute to the organization of instruction, as well as the content and materials used. In Thomas instruction can be identified the van Hieles phases of learning (information, guided orientation, explicitation, free orientation, integration).

The design used in phase III was the same as in phase II. The differences were that Patrik carried out the teaching in another class. A part from the aspects focused on in Thomas class (in phase II), Patrik focused on finding values of a variable for which an algebraic fraction is undefined as understanding the difference and connection between roots of a quadratic equation and factors of a quadratic expression. Besides this, Patrik repeated content discussed in previous lessons at the beginning of each new lesson.

The enacted object of learning in Thomas and Patrik's classes enabled students to discern the process of factorising polynomials and simplifying algebraic expressions written as fractions. In addition, the students had the opportunity to experience: the term cancelling; that factorising is the reverse of the distributive property; both the expressions factor and cancel when working with algebraic expressions written as fractions; to use factorising, cancelling and rules of fraction operations in order to simplify algebraic fraction expressions.

CONCLUSIONS

From the text above, it may be clear that there are many similarities between the aims of the van Hiele levels and the categories found with the help of variation theory, namely, the category A corresponds to level 0, the category B and C correspond to level 1, the category D to level 2, the category E to level 3 and the category F to level 4. The levels and the categories are not situated in the subject matter but in the students' thinking process. The use of these levels and the categories in a teaching learning situation does not stop at the description of "levels of thinking," but seeks to provide a basis for understanding the movement between these levels/categories, and the role of the teacher in assisting such progression. The levels/categories do not give a deterministic view of a fixed progression, but is an empirical description of relatively stable stages and provides guidance on structuring learners' experiences.

The main cause of instrumental thinking in algebra is information overload. There are different levels of thinking and each level has its own network of relations. The transition from one level to the next can be realised by analysing the real critical aspects in students' learning and opening up for dimensions of variation in those aspects. Many properties of thinking can be understood by using the concept of structure. By focusing on the real critical aspects in students' learning and to open up for dimensions of variation in those aspects contribute to a better structure of van Hieles phases of learning, as well as the organization of instruction in the classroom, the content and materials used.

According to van Hiele, progress from one level to the next involves five phases. Each phase involves a higher level of thinking. These phases of learning are significant in providing a framework for instruction aimed to develop understanding of the object of learning to be learned. With reference to the three research questions investigated in this study, it does appear that there is a strong relationship between van Hiele levels and critical aspects. The results of the study are descriptive, revealing aspects of critical aspects in students' learning of how to simplify rational expressions, and contribute to our general understanding about specific efforts to improve the teaching and learning processes themselves.

REFERENCES

Hoffer, A. (1981). Geometry is more than proof. *Mathematics Teacher*, 74, 11-18.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahward, N. J.: Lawrence Erlbaum Associates.

Marton, F., Runesson, U., & Tsui, A.B.M. (2004). The space of learning. In F. Marton & A.B.M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3-42). Mahward, N. J.: Lawrence Erlbaum Associates.

Marton, F. & Tsui, A.B.M. (Eds.). (2004). *Classroom discourse and the space of learning.* Mahwah, N.J.: Lawrence Erlbaum.

Olteanu, C. (2007). *"What could x be?": Second degree equation and quadratic function as objects of learning.* (Doctoral dissertation, Umeå university, Sweden). Retrieved from http://urn.kb.se/resolve?urn=urn:nbn:se:hkr:diva-612

Olteanu, C., & Holmqvist, M. (2009). The Concept of Function - Critical Aspects Induced by Teaching and Textbook. In Callaos, N., Carrasquero, J., V., Oropeza, A., Tremante, A., & Welsch, F. (Ed.) *Proceedings 3rd International Multi-Conference on Society, Cybernetics and Informatics, Vol. II,* (pp. 195-200). Florida: International Institute of Informatics and Systematics. URL: 978-1-934272-71-8

Olteanu, C & Olteanu, L. (2010). To change teaching practice and students' learning of mathematics. *Education Inquiry, 4*(1), 381–397.

Olteanu, C & Olteanu, L. (2012). Improvement of effective communication– the case of subtraction. *International Journal of Science and Mathematics Education*, *10(4)*, 803-826.

Olteanu, C. (2012). Critical aspects as a means to develop students learning to simplify rational expressions. *The 12th International Congress on Mathematical Education (*ICME-12). Seoul.

van Hiele, P. M. (1986). *Structure and Insight. A Theory of Mathematics Education*. London: Academic Press Inc.