ALGEBRAIC THINKING AND FORMALIZED MATHEMATICS – FORMAL REASONING AND THE CONTEXTUAL

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Algebraic symbols allow a reasoning, in which one can detach from the referents of these symbols. This is the starting point for formalized mathematics. Reasoning algebraically on the basis of formalized mathematical objects is a central starting point for school mathematics in the middle- and higher-grades. This paper explores how students reason algebraically with the help of a formal symbolic representation when working on an arithmetic-structural non-routine problem. It is argued that this form of algebraic reasoning has two intermingling aspects, namely formal reasoning and contextual reasoning. This case study suggests that students employ modes of contextual reasoning in order to direct their manipulations of algebraically represented objects.

Keywords: algebraic thinking; algebraic symbols; formalization; formal reasoning

FORMALIZATION IN MATHEMATICS

In the 19th century, algebraic symbols were the central instrument to establish formal mathematics without "ambiguity of meanings [and without] concealed assumptions" (Boyer & Merzbach, 2010, p. 598). Today formalism still plays a strong role in mathematics; it is deeply embedded into the culture of mathematics. Symbolic algebraic language is one of the main tools for formalization. Algebraic symbols "enable us to detach from, and even "forget", their referents in order to produce results efficiently" (Arcavi, 1994, p. 26). Formalization thus lies at the heart of algebraic thinking - it describes those mental acts of individuals, through which a mathematical problem is symbolized in accordance to the rules of the mathematical culture. Formal algebraic thinking is the process by which one attends to this problem with the help of algebraic symbolic language.

Formal algebraic thinking, however, is not equivalent to rule-based manipulating of symbols. Rule-based manipulating may be one aspect of formal algebraic thinking, but there are other aspects as well. I will elaborate on this with the following example, taken from Arcavi (1994, p. 28). Arcavi suggests, that equivalent formal symbolic expressions can be a "possible source of new meanings" (Arcavi 1994, S. 28). He then gives an example for this: "Take an odd number, square it and then substract 1. What can be said about the resulting numbers?" An odd number can be represented by $2n+1$, so that the above sentence can be translated into $(2n+1)^2-1$, which equals $4n(n+1)$. According to Arcavi, this last expression shows, how equivalent expressions can generate new meaning: It uncovers a new property of the resulting numbers, which could not be seen in the original expression: the resulting number must be divisible by 8, because either n or $n+1$ is a multiple of 2.

Arcavi, however, does not address the inherent difficulties, which students may face when they try to see a new meaning in an equivalent expression (such as *4n(n+1)*). First, as Boero points out, a student's transformational activity is guided by processes of anticipation, that is, the student "needs to foresee some aspects of the final shape of the object to be transformed related to the goal to be reached […]" (Boero 2002, p. 100). Second, a student may need a certain mathematical knowledge in order to anticipate a final shape of an algebraic expression of an object. In Arcavi's example, a student needs to realize that one can factorize a number and then find out about this number by looking into its factors. Thus, generating new meaning through looking into equivalent expressions faces an inherent conflict. On the one hand students have to be able to foresee the final shape of a symbolically represented object and, on the other hand, students may generate new meaning only *after* manipulating this symbolically represented object, because only then there are equivalent expressions which denote the same object. But if meaning is only generated afterwards, students' have no means to foresee a final shape of algebraic objects *before* or *while* manipulating these objects.

This paper tries to address and explore this issue of generating meaning during the manipulation of algebraic expressions. This way, it tries to extend the work of Arcavi on processes of meaning making with algebraic symbols, and, by doing so, tries to explore how students deal with objects in their formal algebraic thinking. For that, it discusses two semi-structured interviews, in which students try to solve a non-routine problem with the help of algebraic symbols.

FORMAL ALGEBRAIC REASONING AND CONTEXTUAL REASONING

According to Caspi and Sfard, formalization is a discourse in mathematics with specific meta-rules, which regulate it. These rules are incorporated into the algebraic symbolism of formal mathematics (Caspi & Sfard, 2012). The content of an algebraic expression is a generalization of an arithmetic narrative (Sfard, 2008, p. 120). $a+b=b+a$, for example, represents a general feature of real numbers, and is thus a generalization of an arithmetic structure. Bergsten suggests that formal mathematics allows to address (and solve) not only one problem, but "*types of problems*" (Bergsten, 2008, p. 1). In order to solve types of problems with symbolic, "formal" expressions, one needs an awareness of both its structural and its operational aspects (Bergsten, 2008). Thus, the strength of formal mathematics lies not only being able to address arithmetic generalizations, but in being able to address a pattern or structure, by which one can solve types of problems. For example, a function like $f(x)=ax+b$ represents a pattern between two sets of numbers, denoted by x and $f(x)$. At the same time, $f(x)$ can be regarded as an object itself. With the help of this object, certain types of problems can be solved (those problems in which there is a linear relationship between x and $f(x)$).

Solving a non-routine problem with the help of algebra requires a student to represent this problem with algebraic symbols. For Arzarello et al., this process of symbolizing a situation is a form of condensing meaning, so that students can “grasp the global situation as a whole” (Arzarello et al., 2002, p. 79). Arzarello et al. show that symbolizing is a “game of interpretation”, in which –in a continuous process- more sophisticated conceptual frames are activated, until the student’s “stream of thought […] condenses its temporal, spatial and logical features into an act of thought, […]” (Arzarello et al., 2002, p. 79). The nature of this condensed meaning is relational. A relational meaning is a meaning which is embedded into the relations between the signs and terms of an algebraic expression (Radford, 2009). Relational meaning is opposed to meaning, which is related to the context of the original problem, that is, the original representation of a problem as it is given to a student. The challenge students face when they try to generate relational meaning “[…] is to transform the iconic meaning of formulas into something that no longer designates concrete objects” (Radford, 2009, p. 14). Hence, students’ resources for formal mathematics and for formal algebraic reasoning are the relations between the elements of a given symbolic expression.

Based on these assumptions about the nature of formalization, I want to define formal algebraic thinking. In this paper, algebraic thinking is regarded as analytical reasoning about patterns and structures, where the term “analytic” means to systematically address something through its constituent parts. In line with this, formal algebraic thinking is defined as the reflection upon patterns and structures by seeing/establishing relations between those elements of the symbolic expression, which represent these patterns and structures.

This definition of formal algebraic thinking may help to give insight into how students generate meaning for equivalent expressions. In the example taken from Arcavi, the expression $4n(n+1)$ can be read by establishing relations between it parts. In this expression, $4n$ can be read as a factor. This would suggest divisibility by 4 or 8, depending on n. The expression $n+1$ has to be read in the same way and has additionally to be related to n in $4n$. This (possible) solution illustrates how the structural elements of the symbolic expression $4n(n+1)$ need to have a meaning for the student, which is related to the original problem and its objects, representations and relations.

The definition suggests an answer to the above mentioned conflict of meaning making in Algebra: Based on this notion, meaning making processes in formal algebraic thinking may be moderated by patterns and structures of algebraic expressions. It thus leads to the following question: how do structures and patterns moderate students meaning making processes when they try to proceed from contextual to relational meanings?

METHOD

This exploratory study is based on semi-structured interviews with four groups of two students each. Two groups consisted of 9th grade German high-school students, the other two of 11th grade German high school students. The transcripts presented in this paper are from the two groups of 11th graders, as the ninth graders showed no signs of formal algebraic reasoning. For the question of this paper, this way of choosing probands is acceptable (However, it is quite noteworthy, that there are no signs of formal algebraic reasoning in the interviews with the 9th graders). The transcripts are qualitatively analyzed. The author conducted the interviews. However, he was not the teacher in the aforementioned classes. The interviews were not related to the subject-matter of the current lectures in school.

All of the participating students were introduced to algebra in the 7th grade and had the opportunity to become fluent in algebraic symbolic language at least through the 8th grade. It can also be assumed, that these students have had experiences with certain kinds of formal algebraic reasoning, for example when dealing with two functions in modeling problems or in physics classes, where a formula can be a starting point to deal with certain kind of phenomena like gravity or force. The students were chosen in consultation with the mathematics teacher according to their positive attitude to mathematics and their willingness to participate.

The mathematical task

The students were given a series of similar algebraic non-routine problems called number-triangle ("Zahlendreieck"). All of these number-triangles share the same structure (compare fig. 1); the number in an outer field is the sum of the numbers in the two adjacent inner fields. The numbers in the outer fields add up to the outer sum, which is always given in the rectangle at the bottom right of each number triangle. The number-triangles require the students to think algebraically, as students have to reflect analytically on the triangles' structure. The number-triangles can be solved by a guess-and-check strategy. However, often the students adopt a strategy in which they focus on the structure of the triangle. For example, students will choose a number for a certain field and then search for the numbers of the adjacent fields by applying the rules of the triangle. By solving the first three number triangles in the beginning of the interview, the students get a sense for the structures of number triangles.

Of special interest are the fourth and fifth number triangles (fig. 1). These triangles are designed to trigger formal algebraic reasoning. Other than the previous triangles, the fourth triangle has no solution: the rules of the triangles result in an outer sum, which is always the fixed result of two times the number in the upper inner field added to two times the number in the bottom outer field ($2x+2y$). In case of the fourth triangle the outer sum would be $2 \cdot 7+2 \cdot 9=32$ and not 30. Therefore, the students cannot find a combination of numbers which would solve the fourth triangle in accordance with its rules. When the students get an idea that the triangle cannot be solved by a guess and check strategy, it is expected that students would take a structural perspective on the number triangle. Either on their own initiative or by a silent impulse from the interviewer (giving an x in the left field of the inner triangle), students should begin to reason algebraically in a formal way.

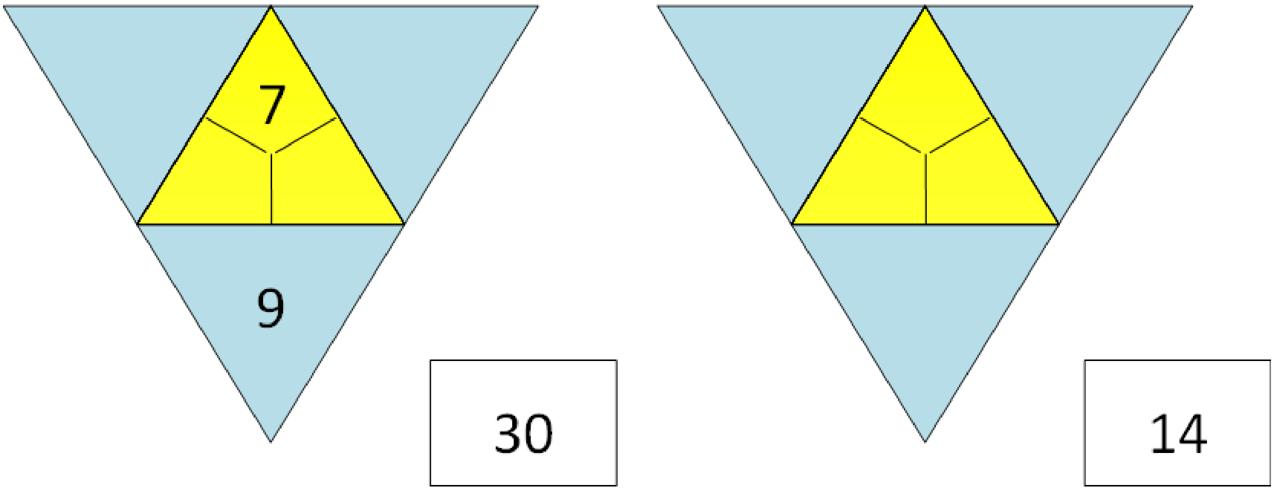

Figure 1: Number-triangles 4 and 5

The following fifth triangle has a given outer sum but is apart from that empty. In this triangle, students may either re-use their symbolic representation of the previous fourth number triangle or adopt a different strategy.

The students have to represent the rules of the number triangles through algebraic symbols, and then act upon this algebraic representation. The students' formal algebraic reasoning is supported by the mental image of the structure of the number triangles, which they gained from their previous arithmetic reasoning on the structure of the first three triangles. It is assumed, that this makes it easier for students to relate their symbolic manipulations to the problem at hand. At the same time, students only need to have fundamental arithmetical knowledge to work on the problem (compared to Arcavi's example, where knowledge about factorization is needed). This should eliminate the problem that students cannot generate a new meaning for an algebraic expression when they lack the necessary knowledge to do so.

As the students in the study are all used to algebraic symbols, it is expected that they can represent the triangle with the help of algebraic symbols. Additionally, they can use this formal representation to reason formally about number triangles, that is, students should see patterns and relations in the algebraic representation. By looking into the patterns and relations of the algebraic representation, students can either identify the problem in the fourth number triangle or, respectively, can find the numbers in the fifth triangle. While the number triangles support students' formal reasoning, they may also act as a possible scaffold for generating meaning while manipulating algebraic expressions. In summary, the five number triangles, which are given to the students during the interview, should allow students to make references to the original features of the number triangles during formal reasoning. Thus, the number triangles provide a scaffold for processes of meaning making during manipulating formal algebraic expressions

FORMAL ALGEBRAIC REASONING AND CONTEXTUAL REFERENCES

In the following section I will discuss transcripts from two groups of 11th graders. I will present how each group of 11^{th} graders tries to solve the fourth and fifth number triangle on the basis of its formal algebraic representation.

Frank and Peters' formal algebraic thinking

Peter and Frank start to solve the problem of the fourth number triangle by trying different number combinations. They systematically check combinations of natural numbers for the lower inner fields with a sum of *9*, e.g. *1* and *8*, *2* and *7* etc. They hypothesize that the outer sum is always *32*. Later, Frank can prove this hypothesis by finding a structure in the triangle:

118	Frank:	[…] Because here we have one 7 *(writes 7 into the upper left field)*, […] here one 7 *(writes 7 into the upper right field)*. This is two times 7, basically *(writes 2·7 above the number triangle)*. And here we have one times 9 (writes 9 into the lower outer field) […] and here *(points at the left inner field)* we have one part, I mean, one x of 9, I mean, times x, I don't know how to put it, say x from 9, and here *(points at the lower tip of the right outer field)* […]
120	Frank:	here it is the rest [of 9].
121	Peter:	x1 and x2 is 9 it is here.

The structure of the triangle helps Peter and Frank to observe, that the outer sum of number triangle can be described arithmetically with *2·7+2·9*. Frank and Peter use variables to describe the composition of the lower outer field. This transcript illustrates how Frank and Peter take a first step towards an algebraic representation of the given number triangle. The use of the variable x (or, x1 and x2) is embedded into the contextual features of the problem, that is, the original representation of the number triangle. This can be seen in the way Frank associates his variable with the according field of the number triangle in line 118 ("x from 9" and accompanying pointing gestures). This suggests that the formalization of the number triangle for Peter and Frank starts within the contextual features of the problem.

When working on the next fifth number triangle, it becomes evident, how Frank and Peter proceed to an algebraic representation of number triangles. Frank and Peter adopt their previous arithmetic representation, but modify it in a way that it fits to the problem of the fifth triangle. The following transcript shows how Frank uses formal reasoning for solving the triangle:

165	Frank:	Yes. We can say we have two times x *(writes this next to the triangle)* ah… plus two times y *(adds "+2y")* equals 14 *(writes "=14")* and for that we can put in any number, y we now can… let's say equals 2, then we would have then 4, then we have 10 left. […]
167	Frank	Half of that is 5 *(writes 5 into the bottom outer field)*. For example.
		[…][interviewer asks them to find the remaining numbers in the triangle]
174	Frank:	It is … it doesn`t matter how to fill them *(writes 2 into the upper inner field)*. Here we can now say 3 and 2 in there *(refers to the two lower inner fields)* […] or 1 and 4 … ah… the values would remain the same.

Frank generalizes the expression $2{\cdot}7+2{\cdot}9$, so that it fits to the new problem. For that, he uses two variables. These variables denote different objects in comparison to the variable in line 118. This way, Frank develops an equation, which describes the fifth number triangle in a formalized way. The general character of this new formula can clearly be seen, as $2x+2y=14$ is a generalization of $2{\cdot}7+2{\cdot}9$. Furthermore, with the generalization comes the notion of the indetermined status of x and y, as "for that we can put in any number" (line 165).

In line 165 there is evidence that for Frank the algebraic expression $2x+2y=14$ is a representation of the number triangle, which allows for formal algebraic reasoning. There are no contextual references being made. Instead, the problem of the fifth number triangle is modified into a problem of finding numbers which solve the equation $2x+2y=14$. The problem of the number triangle is examined on the basis of the relations in the according algebraic expression.

However, there is also evidence that Frank's formal algebraic reasoning relates to contextual features of the problem. In line 167, Frank orients his problem solution towards features of the fourth number triangle. He first fills in a number for the lower outer field. In the previous number triangles, this lower outer field always contained a given number. The same can be seen in line 174, where at first Frank fills in a number into the upper inner field, which was also always given in the previous number triangles. On the basis of these two determined fields Frank now starts to determine the remaining fields of the number triangle with the help of the algebraic representation.

In the interview with Frank and Peter there is evidence which suggests that formal algebraic reasoning may incorporate contextual features. These contextual features remain implicit in the formal algebraic representation. However, as the lines 167 and 174 suggest, contextual features nevertheless influence (or, to some degree, guide) formal algebraic thinking. Hence, it may be plausible, that acting on a problem with its formal algebraic representations ("manipulating") could to some extent be guided by contextual features of a problem.

Stephanie and Laura's formal algebraic thinking

Compared to Frank and Peter, Stephanie and Laura take a different approach to representing the fourth and fifth number triangles. Stephanie and Laura base their formal representation on the notion that the upper inner field and the lower outer field have a ratio of *9* to *6* (which is in fact an overgeneralization, as it is not a feature of *all* number triangles). On the basis of this, they start to represent the fourth triangle with a variable *x*:

151	Stephanie	Here x *(writes x into the upper inner field),* then one has, it is 3/4x *(writes 3/4x into both remaining inner fields),* and down here it would be then 3/2x *(writes 3/2 into the bottom outer field).* […]
153	Stephanie	And then you have to add these two *(points at the two inner fields on the right with x and 3/4x)*

This transcript illustrates how the idea of the ratio 9 over 6 is used to find an algebraic representation. The upper inner and lower outer fields are represented with an *x* and *3/2x* respectively. This suggests that the idea of 9 over 6, which originated from the contextual features of the previous number triangles, is the basis for the algebraic representation. In this representation, each algebraic term (*3/4x*; *x*; *3/2x*) is associated with its according field in the triangle. Hence, the formal algebraic representation is not completely detached from the contextual features of the problem, but remains connected to it.

After a short episode of trying different numbers in the fourth triangle, Stephanie and Laura use the above presented formal algebraic representation to work on the fifth number triangle. The following dialogue shows their first attempt towards a solution:

226	Stephanie:	We have to get 14 *(points at the 14 in the field next to the number triangle)*, the whole over there…
227	Laura:	Ah, yes, 14 divided by 5 is, well…
228	Stephanie:	2.8.
229	Laura:	yes […]
231	Stephanie:	And then, there *(points at the bottom outer field)* has to be 4.2, because it is one and a half. […]"

Stephanie refers to the algebraic representation of the outer sum, which is *5x* (which originates from the sum *(1+3/4)+(1+3/4)+1*). By dividing *14* by *5* Laura can then determine the value of *x*. In line 231, Stephanie uses the algebraic representation of the number in the lower outer field, which is "one and a half" *x*.

This transcript illustrates, that Stephanie and Laura are using their previous algebraic representation to work on the fifth number triangle. It becomes evident, that they are focusing on the factors *3/4*, *3/2* and *1* in order to solve the fifth triangle. This can be seen in the lines 151 and 153. These factors are a somewhat unorthodox way to deal with the formal algebraic representation: They allow the students to focus on the relations of the number triangles. However, at the same time, this representation is contextual, as each factor clearly remains connected to its according field in the triangle. Thus, Stephanie and Laura's algebraic reasoning can be regarded as "formal", as they focus on structures in the triangle with the help of its according algebraic representation. At the same time, the contextual connections of their algebraic representation are guiding their formal reasoning. In the case of Stephanie and Laura, each step in finding a solution for the fifth number triangle is both linked to a field in the number triangle and to the number triangle's "formal" algebraic representation.

SUMMARY AND DISCUSSION

Formal algebraic reasoning and contextual reasoning are connected to the structure of the formal algebraic representation of a given problem. In the interviews there is evidence that meaning is generated on the basis of the relations in algebraic expressions. This process of meaning making is consistent with Radford's findings. On the other hand, the interviews also show how, at the same time, manipulating formal expressions can be informed by contextual relations – structural elements of the original problem are incorporated into manipulating formal algebraic expressions and, this way, into formal algebraic reasoning. Furthermore, formal algebraic representations have (implicit) connections to the original problem. It has been shown, that features of the original problem contribute to the students' formal algebraic reasoning - even though these features are not represented in the algebraic expression or in its relations. Thus, formal algebraic reasoning may be implicitly supported by contextual reasoning: students can foresee the final shape of an object by referring to contextual meaning during the process of manipulating this object. This suggests that meaning making processes in formal algebraic reasoning can remain linked to contextual meaning.

Formal algebraic reasoning may be influenced by the original meaning of a problem (as students see it) and knowledge about features of this problem (e.g. the factorization of a number). Thus, while there is "condensed meaning" (as Arzarello suggests), at the same time there are processes, where this condensed meaning remains regulated by meaning derived from the original problem and its representation. Hence, the conflict of generating meaning during the manipulation of algebraic expressions may be resolved by acknowledging, that, in the process of manipulating, the anticipation of the final shape of an object is adjusted and regulated by contextual reasoning.

Formal algebraic reasoning is a central part of mathematics in the higher grades. It is relevant for solving non-routine problems or for modeling real-world situations (in analytical geometry, in infinitesimal calculus) or even for handling complex concepts in physics or chemistry with the help of symbolic algebraic representations. This study may provide a plausible explanation for some problems students have in formal mathematics. In situations, in which students have difficulties to see contextual features of a problem (e.g. if they are not made explicit or are not directly visible), students may have problems to direct their formal algebraic reasoning, as they cannot foresee the final shape of an object. For example, addressing the divisibility of a number by factorizing it, like in the example taken from Arcavi, may not be a self-evident contextual relation for students, and - when this relation is not being made - may hinder the students to anticipate the final shape of an algebraic object, towards they could have aimed their manipulations of the initial object.

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