DIFFERENT WAYS OF GRASPING STRUCTURE IN ARITHMETICAL TASKS, AS STEPS TOWARD ALGEBRA

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In this study we propose an analysis of some interesting solving processes of 9-grade students engaged in an arithmetical task. It originates as an item of a national test that has proven to be very critical for Italian students. By changing the way of administering the task, and also by virtue of some interviews, we got the opportunity to observe interesting students' behaviours, some of which throw, in our opinion, new light on students' sense-making processes in the borderline between arithmetic and algebra.

KEYWORDS: *arithmetic-algebra, sense-making*

INTRODUCTION

In these last years new perspectives about the teaching and learning of algebra are emerging, due, among other things, to the impact of educational technologies (see e. g. De Vries & Mottier, 2006) and to the achievements of Early Algebra (for a wide overview, see Cai & Knuth, 2011). Moreover, interesting hypotheses about the presence of a sort of algebraic discourse within the daily life media have been advanced: "*It is possible that these days algebra is simply 'in the air'* […]. *With the help of media, algebraic forms of expression may even be infiltrating colloquial discourses*" (Caspi & Sfard, 2010, p. 255-256).

Our research group has been working for several years on the teaching and learning of algebra with particular care to motivational aspects and sense-making processes (see e. g. Guidoni, Iannece & Tortora, 2005). We have been studying these processes mainly adopting a Vygotskian research perspective, that is arranging suitable class settings, where social interaction and immersion in culturally relevant activities are enhanced, at a secondary school level (Iannece & Romano, 2008), as well as at a primary one (Mellone, 2011), where we share the basic claims of Early Algebra.

In this paper, of an exploratory character, we examine some specific students' behaviours, that we observed within a didactic activity, devised for understanding and analysing the poor performances realized in some items of the annual Italian national assessment for 10-grade students organized by INVALSI (*Istituto Nazionale per la VALutazione del Sistema educativo di Istruzione e di formazione*) in the year 2010-2011 (a complete account of the test, together with an analysis of students' difficulties can be found in http://www.invalsi.it/snv1011/documenti/Rapporto_SNV%202010-11_e_Prova_nazionale_2011.pdf).

Some poor results of the test come as no surprise, rather they confirm some of the most frequently observed difficulties met by students. But the behaviours we have noticed seem to suggest, in our opinion, new research questions and lines, as we will try to show in the sequel.

The following (D16) is one of the INVALSI test items, where the Italian students encountered major difficulties, as shown in the table below:

D16. The expression $10^{37} + 10^{38}$ is also equal to:

A. 20^{75}

B. 10^{7}

C. $11 \cdot 10^{37}$

D. $10^{37 \cdot 38}$

A (%)	B (%)	C (%)	D (%)	No answer (%)
35.0	1.9	22.0 (correct)	38.7	2.4

This result, together with the results to the whole test, acted on us as a stimulus for trying to understand more deeply their causes. Therefore, we have arranged several slightly different modalities for administrating the questions to different students. Moreover, in a second phase, a selected sample of them have been briefly interviewed. We have collected several data and are now just interpreting them, as the object of a deep study. Here, we want to focus on a few episodes referred to students' answers for this item, mainly to underline how, when the students are (more) free to select and to express their own strategies, and their tasks are examined with new eyes, the spectrum of their behaviours considerably enlarge and often goes beyond any simplified classification attempt. This may even suggest new reflections and ideas on the usual meaning of the algebraic notions and procedures.

THEORETICAL FRAMEWORK

Algebra is a mathematical domain in which the search for meaning is very problematic. Indeed, many are the meanings that can be given to the algebraic symbols and to the word 'Algebra' itself, in the official Mathematics still before than in mathematics education. Among these, the link with arithmetic is undoubtedly the most deeply rooted in the tradition, whereby algebra is commonly understood as the language for expressing general statements about numbers. But in the history a crucial turning point occurs at the beginning of the last century: this is best expressed by Bourbaki's own words, that convey the fervour for the new discoveries and approaches: "*It is no doubt the possibility of these successive extensions, in which the form of the calculations remained the same, whereas the nature of the mathematical entities subjected to these calculations* *varied considerably, was responsible for the gradual isolation of the guiding principle of the modern mathematics, namely that mathematical entities in themselves are of little importance; what matters are their relations.*" (Bourbaki, 1974, p. xxi). It is just there that we can find the ultimate roots of the emphasis given in the last decades by all world school curricula to the syntactical aspects of algebra and the corresponding tendency to leave meanings in the shadow.

However, it is widely acknowledged that from a didactic point of view the link between arithmetic and algebra is the high road to support and justify the introduction and the development of algebraic skills. Therefore, the problem becomes rather to understand the different aspects of this link and to suitably manage them by means of an effective didactic mediation, in order to avoid the common and well known difficulties met by students.

In this direction, many research lines suggest that the link arithmetic-algebra cannot be reduced to a simple one-way path (see, for instance, the studies inspired to Davydov's ideas (Davydov, 1982), like e. g. (Iannece, Mellone & Tortora 2010)). In a recent study (2011), Subramaniam and Banerjee document that as early as in the twelfth century famous Indian text 'Bhaskara', the role of algebra is explicitly viewed as a foundation rather than as a mere generalization for arithmetic, along a typical two-ways relationship: algebra "*was viewed both as a domain where the rationales for computations were grasped and as a furnace where new computational techniques were forged*" (Subramaniam & Banerjee, 2011, p. 95).

Turning to students' behaviours, the international literature about the solution of algebraic exercises and word problems offers several examples that suggest an apparent 'suspension of sense-making' (see for example Schoenfeld, 1991). On the other hand, according to wide evidence coming from neuroscience (see for example Rizzolatti & Sinigaglia, 2006), there doesn't exist in our brain an absolute *absence of meaning*, but instead the brain's way of working is always subjected to an automatic and sometimes unconscious dynamics of search for meaning. In particular, whenever the time at our disposal is quite short, what often happens is a suspension of the meanings that come from mature knowledge in favour of an appeal to previous more basic resources or to rapid analogical reasonings.

So, along a solving process, students' making-sense could be genuinely a search for solutions plausible and coherent with the text of the exercise, typically *inside* the discipline (in this situation different rationalities could even be recognized, see for example the use that Boero & Morselli (2011) make of Habermas' theory about different rationalities occurring in the execution of mathematical tasks). Or, students can also look for a, so to say, *outside* sense, basing their interpretation of the task on the fact that who assignes it is an adult bearing an authority, and so trying to interpret this adult's behaviour too: "[…] *there is a* *gross mismatch between the goals that the teacher thinks he or she is getting for students and the goals that students actually seek to achieve. In other words, the teacher believes that the students are operating in a mathematical context when their overall goals are primarily social rather than mathematical in nature*" (Cobb, 1986, p. 8). For example, during the solving process of an algebraic exercise, it can happen that its sense be identified with the mechanical use of formulas or algorithms, because students' perception is often that the meaning of such an exercise is exactly to show that some rules have been well memorized. This is also the effect of the belief that the success in mathematics strongly depends from the ability to execute a task in a brief time, belief that often comes from the first grades of school and is supported by the many test that students meet during their lives. As claimed in (Arcavi, 2005), the kind of sense making followed by students in their solving process is strongly linked to the class culture. Therefore, the classroom culture has a central role in behaviour or habit of the learners and this has important implications in the didactic practice. Arcavi gives some advices in what direction the didactic practice could move, he suggests *"asking students to develop the habit not to jump to symbols right away, but to make sense of the problem, to draw a graph or a picture, to encourage them to describe what they see and to reason about it"* (Arcavi, 2005, p. 45).

METHODOLOGY AND CONTEXT

The INVALSI test for grade 10 lasts 90 minutes and contains 30 questions, in most cases multiple-choice questions, some of them with several items. The test is administered to all the Italian students and some of the questions turned out very critical for them, as we have already said. So, we decided to involve our group of teachers in a research work in order to deepen the reasons of these poor performances. We are convinced that the formats of the questions (multiple-choice, above all) and the short time have greatly contributed to students' failure, triggering a sense-making oriented to a "social survival" (Cobb, 1986) – this term being used to refer to the general ability of facing complex human interaction and constraints depending from various intentions and goals, – more than one genuinely linked to the discipline.

As a first step of our work, each question and the corresponding national results has been carefully studied. Then we decided to modify the way of administering some selected critical items (one of them is D16) and to test them in several classes of different grades. The teachers inserted the selected questions in their monthly written tests, which usually include about 15 exercises of several kinds (open questions, closed questions, problem solving), and the students were asked to justify their answers, even for the multiple-choice questions. The time allowed for answers is generally longer than in INVALSI test. In a few cases a student has been interviewed, after the correction of the classwork, in order to better understand some passages in her/his text or to realize to what extent was (s)he aware of the legitimacy of the procedure employed. The general purpose was to reflect on students' solution processes and, possibly, to link them with the didactic strategies of the teachers. In fact, it is worth noticing that the educational approach of the teachers of our team is inspired by Vygotsky's ideas about the social and semiotic nature of learning. In particular, they often approach algebra by means of numerical problems organized around the cycle Conjecture-Argumentation-Proof: firstly, students are asked to search for regularities and to make conjectures, later they are encouraged to discuss and finally to provide a mathematical proof or a counter-example to the conjectures. Often proofs need new algebraic techniques that, in this way, aren't proposed as mere theoretical notions but rather as suitable tools to make generalizations, tools that only afterward will be stabilized as standard procedures to be used in similar problems.

In this paper we report an analysis of students' answers to the D16 item. As the item involves tasks usually faced in the 9th grade, we decided to administrate it in May, 2012 to 43 students of two 9-grade classes.

D16 is very interesting for many reasons. First of all, it is a context-free task that can be approached through different solving strategies: for example, a) it can be solved comparing the magnitudes of the numbers; b) the particular base 10 of the powers could induce students to perform arithmetical computations; c) the sum of two powers with the same base can wrongly induce to apply powers properties, as suggested by the two distractors A and D (38% of Italian students chose D); d) the expression can be manipulated as an algebraic one, using a symbolic rule like factorization to facilitate a complex arithmetical computation, becoming in this way a good example of interplay between algebra and arithmetic. Moreover, D16 requires a transformation of an arithmetical expression: according with (Subramaniam & Banerjee, 2011), working with the operational composition of numbers is one of the key idea marking the transition from arithmetic to algebra.

The following table contains the answers to question D16 in the two classes:

Choice	A	B	C	D	No answer
Number of students	4	0	28	11	0

Even if our sample is not representative and of sufficient size to show statistical significance, we notice that the percentage of correct answers is considerably higher than the national one (28/45 ≅ 62%, vs. 22%). In the next section we will analyse a few protocols, some of which contain peculiar reasonings and solving processes, that can be seen, in a sense, as alternative to the usual arithmetical/algebraic procedures.

ANALYSIS OF SOME PROTOCOLS

First of all, let us say that many of the students give correct answers to the question, probably thanks – although we obviously have no direct evidence of this – to the extra time allowed to them (with respect to the official test), and to the teacher's care in giving sense to the transition arithmetic-algebra. Secondly, many correct answers employ standard algebraic procedures (which are also those expected by the authors of INVALSI test), as well exemplified in Lucia's protocol (Fig. 1), where the result is obtained using a factorization step, carefully expressed in algebraic terms and applied to the numerical case.

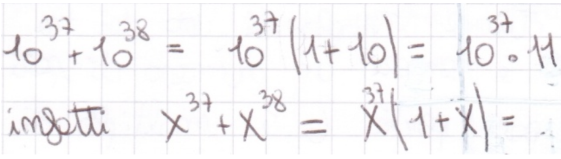

Fig. 1 - Lucia's protocol

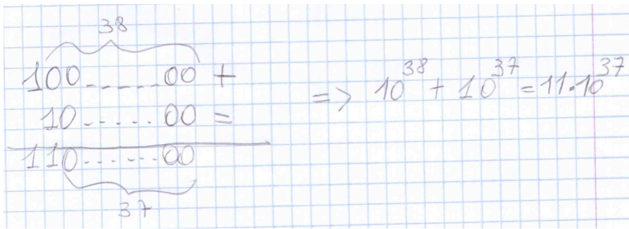

Fig. 2 - Ciro's Protocol

But what is more interesting for us is to try to understand the less obvious-linear-"normal" ways followed by other students, both in the cases they succeed and in the cases they don't.

Ciro's (Fig. 2), Rosanna's (Fig. 3) and Giuseppe's (Fig. 4) excerpts are similar, in that they catch the right answer using procedures that have some algebraic features, but do not abandon the arithmetical domain. Another common characteristic is that their results are, from a strictly logical point of view, just likely, maybe strongly likely, but not safely proved. Or, at least, it can be said that possible refinements of their reasonings, to get rigorous proofs, are not a priority for them.

The right answer is C. In fact, if for example we figure 10^2 + 10^3 out, we get 100 + 1000 = 1100. The other options are wrong, since they look as if the properties of powers had been applied to the expression 10^{37} + 10^{38}, whereas they can only be applied to multiplications or divisions.

Fig. 3 - Rosanna's protocol (our translation)

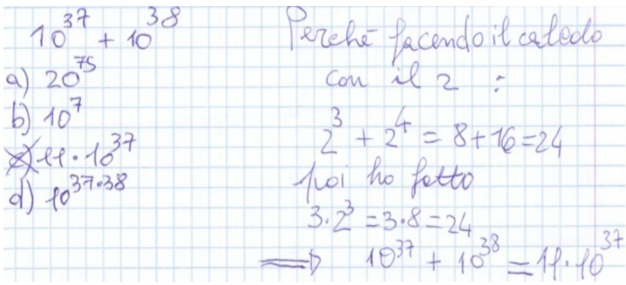

Fig. 4 - Giuseppe's protocol. He says: "Since operating with 2… then I have made…"

Ciro's way of avoiding heavy computations is as easy as an egg of Columbus and reveals a strong confidence with familiar decimal expansions and related algorithms. The use of ellipsis in his decimal writing of powers of ten is a clever shortcut, that may be considered as a kind of algebraic attitude, if we intend algebra not as generalization but as focus on structure, as pointed out in the theoretical framework. In fact, Ciro is able to recognize his result as the product $11 \cdot 10^{37}$, where we don't consider so important whether the presence of this answer among the possible options helped him or not. Moreover, we guess that the path toward forms of generalizations is wide open for Ciro: simply delete the numbers 37 and 38 on the braces and that's it (of course some letters should be introduced on the right side).

Rosanna's strategy is not so different, but she prefers a simplified numerical example to justify her choice. She notices that the numbers 37 and 38 are consecutive and is quite aware of the importance of this fact: again a structural aspect underlying the specific numerical case. And she is able to substitute two simpler numbers, that is 2 and 3, to see and to explain the result for the larger ones using the numbers chosen as a so called *generic* example. Moreover she skillfully detects the deceiving role of the distractors, even if we wonder whether she uses her metacognitive attitude to rely more firmly on her result or to provide a more complete answer.

Giuseppe's protocol is quite complex, and perhaps even more interesting. He renounces to the advantages of decimal representation of numbers and prefers to rely on smaller numbers and on his familiarity with operations on them. The choice made by Giuseppe brings him farther from the arithmetical reasoning, requiring a quite rich algebraic treatment of the problem. But, as a matter of fact, he succeeds in the task of controlling two simultaneous substitutions (the base 2 for the base 10 and the exponents 2 and 3 for 37 and 38), grasping the essential (structural) relationships, and succeeds in correctly applying the obtained results to the original numerical case. Here Giuseppe, like Rosanna, apparently uses a *generic* example. But his direct jump from a single case to another one without the interposed support of a general argument, turns out to be a bit reckless. A deeper insight on his way of reasoning comes from the interview, whose essential passages are reported here.

Interviewer: Why did you change 10^{37} and 10^{38} into 2^3 and 2^4, instead of computing the two powers of ten?

Giuseppe: I chose smaller numbers to make easier calculations.

I.: But how just 2 and 3 as exponents?

G.: Since 37 e 38 are consecutive numbers, so I selected two consecutive numbers, but very small.

I.: Ok, and then?

G.: Well, $2^3 + 2^4$ is 24, then I noticed that, in the answer C, 10^{38} becomes 11 [sic!], therefore I transformed 2^4 into 3, computed $3 \cdot 2^3 = 24$, and since the results turn out to be the same, I chose C.

I.: Oh, but you certainly remember what you have heard several times, that one example doesn't make a proof! How could you be sure of your answer, after only one example?

G.: Because I controlled the other answers.

I.: What do you mean?

G.: I computed 4^7 [Giuseppe points on the option A on his sheet], 2^7 [on B] and 2^{12} [on D] and none of them resulted in 24, then I guessed that if only the option C works for 2, then the same must hold true for 10.

As can be seen, Giuseppe's justifications of his procedure confirm what has already been deduced from his protocol, that is a solid grasping of the structural features of the numerical expression. Indeed he handles the arithmetical expressions occurring in the various answers as algebraic ones, substituting time after time the numbers of his example. But what is more interesting is the intertwining between his reasoning toward the right result and the way he uses the distractors, in particular the hypothesis that only one of the answers is correct. His mathematical (or better logical) argument is frankly convincing although quite complex, to the point that we confess some difficulty of ours in transforming it in a safe logical argument. In other words, we could say that Giuseppe's argument is not certain, but only a strongly reliable one.

Mariarosaria's protocol (Fig. 5) shows a typical example of a wrong choice: she is attracted by the option D, the most preferred by Italian students.

$10^{37} + 10^{38}$ can be written as $10^{37 \cdot 38}$ since the exponents aren't equal, therefore we cannot concentrate on equal bases, there isn't a multiplication, so we can't add the exponents, but with equal bases we multiply the exponents even if there is a sum of the bases

Fig. 5 - Mariarosaria's protocol (our translation)

Also M. was interviewed, to better understand her strategy and in particular why she completely disregarded the right answer C.

I.: Why didn't you consider C, when you realized that something didn't work with the power properties?

M.: Well, the other options looked as more likely; C was too strange.

Perhaps there is not so much to comment on this: the distractor hit. But something can still be deepened and some 'reasonable' insights identified in Mariarosaria's words. First, the word "concentrate" belongs to the class language (and therefore to classroom norms and practices, in the sense of (Cobb, 1986)), as witnessed by the teacher: when she give instructions about products or divisions of powers, she usually says: "when the exponents are equal, you must concentrate on bases, when bases are equal, on exponents". Apparently, teacher's message was received, but the emphasis on the two possible cases caused neglecting that only the multiplicative structure is involved. Anyway, it can be said that M. views the exercise as concerning just calculation techniques of powers (she calls "strange" the option C) to the extent that she dares to invent a new rule to justify her choice.

CONCLUSION

First of all, we want to emphasize that the Vygotskian methodology used by the teacher and in particular the continuous attention devoted to students' reasonings is the unvaluable condition that allows, here as well as in general, the students to freely express their views and the teachers and researchers to observe them and to achieve new insights about students' behaviours and their sense-making processes.

In our case, the various students' attempts to solve the task appear to us hardly classifiable, using usual tools. This raises, in our opinion, a lot of didactic and research questions. Let us focus on two of them. First, which kind of rationality can be recognized in, say, Giuseppe's argument (but, to some extent, also in other students' arguments)? We recall here the already quoted deep analysis of students' behaviours that utilize Habermas' subtle distinctions of various kinds of rationality (Boero & Morselli, 2011) displayed by students engaged in mathematical tasks. But we hypothesise also that some kind of 'social' rationality is active (Cobb, 1986). The second question is if it is better to encourage or to discourage this kind of arguments, and moreover if we are disposed to consider them genuine mathematics.

Of course, we have no definite answer to these questions, but are convinced that they are more and more urgent, since multiple choice tests are just examples of problematic situations very frequent in today life. We mean that the diffusion of complex multimedial tools requires a great capacity to take rapid decisions, supported by solid arguments that, however, almost never can be completely sure. To face these problems is, in our opinion, one of the biggest challenges of tomorrow educators.

Finally, we want to point out that the variations in the way of administrating the task were also used by teachers involved in this research as a tool to reflect on their own didactical practice. For this reasons we are convinced that this kind of experience could provide suggestions for a possible wide scale work with teachers, both to support them in facing the national test, and as a good hint for in-service training.

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