LIMIT OF THE SYNTACTICAL METHODS IN SECONDARY SCHOOL ALGEBRA

Rahim KOUKI

LAMSIN ENIT de Tunis, Université de Tunis El Manar– Tunisia

Faïza Chellougui

Faculté des Sciences de Bizerte, Université de Carthage – Tunisia

One of the main difficulties that secondary school students have to face in Algebra is the insufficient mastering of syntactic rules as often pointed out both by teachers as well as researchers. In this paper, we adopt a logical point of view on equations and inequations and we support the hypothesis that for an adequate appropriation of theses two notions, it is necessary to be able to articulate syntax and semantic. We start by explaining what is meant by "a logical point of view". Then, we examine in which respect the dialectic between syntax and semantic appears in Tunisian textbooks, through an analysis relying on both the dialectic between syntax and semantics as well as mathematical praxeologies. Finally, we provide an example enlightening the paramount importance, in some cases, of the semantic point of view in order to solve (in)equations.

INTRODUCTION

One could think that the main challenge in the teaching and learning of the resolution of (in)equations in high school concerns algebraic rules. However, it is not always the case that applying such rules ensures effectiveness, as we will see in the example that we present in the remainder.

In this paper, we adopt a logical point of view on (in)equation, and we support the hypothesis that, for an adequate appropriation of (in)equations, it is necessary to be able to articulate both syntactic and semantic aspect. We first briefly present this approach referring to predicate calculus. Then, we examine in which respect the dialectic between syntax and semantics appears in Tunisian textbooks, by analysis relying on both the dialectic between syntax and semantics aspect as well as mathematical praxeologies Chevallard (1998). Finally, we provide an example enlightening the fact that, in some cases, the lack of mobilization of the semantic point of view may hamper students of successfully solving (in)equations.

A LOGICAL POINT OF VIEW ON EQUATIONS AND INEQUATIONS

Chevallard (1989) emphasizes the essential dialectic between Arithmetic and algebraic calculations that he interprets as a link between syntax and semantics. Indeed the author explains that "when, in the sixth class, the teacher moves from $2+3=5$ and $3+2=5$ to the general relationship $a+b=b+a$, he moves from computing on numbers (integer) to an algebraic calculation (integer coefficient). In other words, an algebraic calculation, that we do not define more precisely here, makes a clear syntax to which the associated computational domain provides a semantic" [1] (Chevallard, 1989, p. 51). Moreover, the author shows that the students' relationship to algebraic calculation doesn't incorporate the idea of a dialectics between manipulation of algebraic expressions and substitution of numerical values in these expressions.

Furthermore, authors such as Selden & Selden (1995), Durand-Guerrier (1999, 2003) Durand-Guerrier & Al. (2000) Durand-Guerrier & Arsac (2003, 2005), Chellougui (2003, 2009), Weber & Alcock (2004), Ben Kilani (2005) and Iannone & Nardi (2007) have pointed out the relevance of the logical point of view for the analysis of mathematical reasoning in an educational perspective, mainly in calculus.

As for us, we consider that the issue of linking both perspectives, i.e, semantics and syntax, has not been extensively addressed in research, neither in research on the teaching of mathematics in general, nor in the teaching of algebra in high school.

In our research program, a main question addresses the possibility of identifying in the development of concepts of equation and inequation, phenomena related to the dialectical syntax / semantic. In order to tackle this issue, following Durand-Guerrier (2008), we make the assumption, that the semantic conception of the truth, developed by Tarski (1936, 1944 and 1960) through the notion of satisfaction of an open sentence by an element of the domain of interpretation, provides a relevant framework.

In formalised languages, as the predicate calculus, a formula which is not a logical theorem or a contradiction is neither true nor false. In order to provide its truth value, it is necessary to choose an interpretation, namely a domain of objects, properties and/or relation. Further on it is an important fact that a formula may be interpreted to be a true statement in one interpretation, and to be a false statement in another onee. In the case of a logical theorem, it is true whatever the interpretation, or a in case of a contradiction, it is false whatever the interpretation is.

For example, let us consider the formula "$\forall x \exists y F(x,y) \Rightarrow \exists y \forall x F(x,y)$ and the following interpretation: the domain of objects is the real number set; F is interpreted by the relationship "$\leq$" the statement that interprets the formula is "$\forall_R x \exists_R y, x \leq y \Leftrightarrow \exists_R y \forall_R x, x \leq y$"; the antecedent of the implication is true, and the consequent is false, so the statement is false. Nevertheless, if we consider an upper-bounded part of the real number set, then the corresponding statement is true.

While introducing the notion of satisfaction of a formula, Tarski explicitly refers to the notion of mathematical equation. In this perspective, an equation is an open sentence; given a domain, it may be true for some elements of the domain, but not all, or true for every element of the domain, or false for every element. Solving an (in) equation comes to determine the elements satisfying this open sentence. In addition, it is possible that, given an equation, no solution fits in the domain, and one or more than one solution in another domain. For example "$x^2+1=0$" has no solution in the real number set, but two solutions in the complex number set.

On the other hand, in formalised languages, syntax is the term used in logic in a broad sense including

1. The study of the rules of well-formedness of expressions of a given language (the grammar);

2. The set of rules of derivation in an established theory of demonstration in the formal sense of the term, opposed to the semantics, which takes into account the interpretations.

For example, two equations are equivalent if and only if:

1.They are satisfied by exactly the same elements (semantic point of view);

2. It is possible to transform one equation into other one applying algebraic rules preserving equivalence (syntactical point of view). It is important to notice that some algebraic rules do not preserve equivalence, so that in some cases, it is necessary to come back to the involved domain for a semantic control.

For example, let us consider the equation "$x^2-2=|3x+2|$" whose set of solutions in the real number set is $\{-3,4\}$. By applying syntactic rules and transformations without a semantic control $\{-3,-1,0,4\}$, could be considered as the set of solutions although it contains elements that do not satisfy the equation.

Finally, logical semantics allows us to give precise definitions of (in) equations on the one hand, and supports our claim of the relevance of taking into account both semantic and syntactic aspects in mathematical reasoning.

III. CROSSING MATHEMATICAL PRAXEOLOGIES AND LOGICAL SEMANTICS FOR A STUDY OF TUNISIAN SYLLABUS AND TEXTBOOKS

In compliance with Chevallard (1998)'s point of view on didactic transposition, and in order to identify the institutional prescription concerning the knowledge to be taught on (in) equations, we have analysed the Tunisian syllabus and textbooks, through the mathematical praxeologies (Chevallard 1992).

According to Chevallard, all human activities, and namely, mathematical activity, can be described through praxeologies. A praxeology is composed of two blocks. The first block is named the *praxis*, that refers to the practice, and has two components: *Type of the task*: what is to be done (e.g., solve a quadratic equation in complex numbers set) and *Techniques*: the ways to achieve a certain type of task (e.g., the compute discriminator, study its sign; apply the relevant formula to get both solutions). The second block is named the *logos*, which refers to the theory, and also has two components: the *Technology* that is a discourse able to justify a technique (i.e. write down and factorize the canonical decomposition of the quadratic trinomial) and the *Theory*, which provides a justification for a technology (e.g., the axioms and properties of the complex numbers field).

However, we consider that this theoretical perspective is not sufficient to grasp the dialectics between syntax and semantics. Thus, we have enriched the categorisation by crossing praxeologies (namely the praxis [2]) with this dialectics between syntax and semantics. In addition, we take into account the registers of semiotic representation (Duval, 1991) that play a crucial role in algebra and, following Robert (1998), we consider the way in which students work on notions in exercises or problems (merely applying them or being able to mobilize them when required, or on their own). This led us to elaborate the bi-dimensional grid sketched by Table 1.

Technique /Task [3] **NMFK** [4]	***Semantic***	***Syntactic***	**Hybrid** [5]
Elementary	Verify / numerical-graphic	Factorize / algebraic	Solve/numerical-algebraic
Mobilized	Interpret/graphic-algebraic	Demonstrate/Analytic	Study and represent / algebraic-analytic-graphic
Available	Existence/analytic-graphic	Discuss/algebraic	Conjectur/numerical-algebraic-graphic

Table 1: Bi-dimensional grid for analysing program and textbooks

At a glance, our study has shown that

- The treatment of (in)equations, in exercises and in problems of synthesis, requires mostly the use of syntactic techniques, whereas, introductive activities, that are often modeling's problem or graphic situations, take significantly into account the articulation between syntax and semantics.
- The relationship between a Cartesian equation of a curve (e.g. a conic given by a non necessarily functional relation "$R(x, y) = 0$") and a graphical representation of a function (e.g., quadratics) $y = f(x)$ is not clearly stated, the conics appearing mainly, if not exclusively, as a graphic representation of functions.
- Introduction of functions mainly relies on semantics techniques: substitution values to variables, interpretation, etc. (Chevallard, 1989)

DIDACTIC INVESTIGATION

As part of a research carried out within our PhD activities, we have submitted a questionnaire to secondary school students on the one hand, and tertiary students [6] on the another hand [7] (cf. Kouki 2008).

Through this survey we were interested in assessing which point of view (semantic and / or syntax) was preferably mobilized by the students in solving (in) equations. In addition, we tried to identify their ability to smoothly move from one or more registers of semiotic representation to another one.

The mathematical and didactic analysis of different strategies of resolution in the treatment of the various tasks were based on theoretical frameworks crossing the logical semantic perspective and the didactical anthropology of the didactics (Chevallard 1992), and taking in account the system of semiotic representation and registers, for analysing students' production consisting in signs, graphics and algebraic writings (cf. Duval 1991).

For this analysis, we refer to the grid (cf. Table 1) and we assign a code to each type of technique throughout the questionnaire: each type of technique has been denoted by t_{a-b} , indexes a, and b stand respectively for:

- a : The logical categorization of the technique in terms of syntactic, semantic or mixed respectively denoted "se", "sy" and "H".
- b : the classification of a technique in a register of semiotic representation of the type graph, numerical, algebraic, analytical or functional, etc. respectively denoted grph, num, alg, and anl, etc.

The overall results of the questionnaire analysis lead to the conclusion that students use preferably syntactical techniques as soon as they are available, even in cases where intermediate questions involving semantics treatments in numerical or graphical register were introduced. On the other hand, we have observed a fairly high percentage of pupils and students who use semantics tools of resolution only in cases where such tools are explicitly required by the given task. Moreover, it is worth to mention the difference between the type of procedures (syntactic/semantics) for the resolution of exercises of the questionnaire between students at the same level makes us think that it could be linked to the practice of classroom. This agrees with Alcock results at the tertiary level (Alcock 2009). Further researches, facts are in compliance to explore this issue.

We chose to present here the results provided by the analysis of an exercise out of our questionnaire that emphasises the fact that in some cases, the use of purely syntactic methods hampers from solving the given task. Indeed, the latter task requires adopting a semantic point of view articulating (in)equation and curves.

The exercise looks for retrieving all the solutions of the "product inequation" in two real variables $(y-x)(y-x^2+3x)>0$.

The difference in level of pupils and students led us to propose the exercise in two different ways.

Regarding the second group of students of secondary and third year, the exercise that has been proposed is a following:

Let f and g functions defined over IR by: $f(x)=x$ and $g(x)=x^2-3x$.

Let Γ_f and Γ_g the graphs of f and g live in an orthonormal $(O,\vec{i},\vec{j})$.

1) Representing Γ_f and Γ_g in Cartesian coordinates system $(O,\vec{i},\vec{j})$.
2) Determine the sign of $h(x,y)=(y-x)(y-x^2+3x)$ in each of the following pairs:
   - $(x,y)=(2,1)$
   - $(x,y)=(1,3)$
   - $(x,y)=(5,4)$
   - $(x,y)=(-2,-1)$
   - $(x,y)=(-1,-2)$
   - $(x,y)=(6,7)$
1) Place in $(O,\vec{i},\vec{j})$, points A,B,C,D and F respective coordinates of: $(2,1)$, $(1,3)$, $(5,4)$, $(-2,-1)$, $(-1,-2)$ et $(6,7)$.
2) Determine by calculation or graphically, all solutions of the inequation $(y-x)(y-x^2+3x)>0$.

The first three questions were proposed to provide indicators that could lead students to move to algebraic graph's register, to represent the different parts of the plane determined by the two graphic representations Γ_f and Γ_g, and to make a correspondence between the sign of each values of the function h in different points A, B, C and D in each region, on the other side.

Regarding the students of "Preparatory Classes of engineering" we have considered that they had acquired sufficient ability to solve the inequation based solely on graphs Γ_f and Γ_g.

Consequently, we have chosen to propose the exercise as follows:

Let f and g functions defined over IR by: $f(x)=x$ and $g(x)=x^2-3x$.

Let Γ_f and Γ_g the graphs of f and g live in an orthonormal $(O,\vec{i},\vec{j})$.

1) Representing Γ_f and Γ_g landmarks in $(O,\vec{i},\vec{j})$.
2) Determine by calculation or graphically, all solutions of the inequation $(y-x)(y-x^2+3x)>0$.

The *a posteriori* analysis of the first three issues has shown that students can mobilize different types of technical records articulating different algebraic numerical and analytical graphics to meet the intermediate spots.

In this paper, we put the focus on the resolution of the inequation $(y-x)(y-x^2+3x)>0$, which can be solved, straightforwardly, by the technique of a change of register: From the register of graphic writings algebraic framework of algebraic equations involving the straight line $y-x=0$ and the parabola $y-x^2+3x=0$ to graphs Γ_f and Γ_g which intersect at the origin of the landmarks orthonormal $(O,\vec{i},\vec{j})$ and the point of coordinates $(4,4)$. The line divides the plane into two half planes with open upper half plane is the points $M(x,y)$ as far as $y-x>0$and the interior of the parabola is the set of points $M(x,y)$ as $y-x^2+3x>0$. The sign of the relationship $(y-x)(y-x^2+3x)$ is determined by Table 2.

Sign

Regions					
	-	-	+	+	+
	-	+	+	-	-
	+	-	+	-	-

Figure

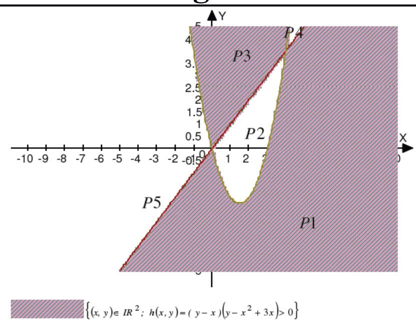

Table 2: Sign of the "product inequation" in different parts of the plan

In conclusion, the set of the solutions of the inequation is the set of pairs that exactly correspond to the coordinates of the points in the region $P_1 \cup P_3$.

The *a priori* analysis of this question showed that three types of techniques could be used by students; we respectively denote these techniques T_1, T_2 and T_3.

The first technique T_1 is of type "Hybrid"; it consists in interpreting graphically the sign of the two algebraic expressions "$y-x$" and "$y-x^2+3x$" and in concluding that the product is strictly positive in the region $P_1 \cup P_3$.

The second technique T_2 is of type "semantics"; it consists in linking graphic and algebraic register, affecting the sign of the factors for each point according with its position in the Cartesian plane and then deducing which are the points in the region .

The third technique T_3 is a purely syntactic one in algebraic register that consists of attempts in order to transform the inequation $(y-x)(y-x^2+3x)>0$. This technique appears to be inoperative at the considered level.

The analysis of corpus of 143 copies [8] of pupils' and students' work shows that in the first three questions to which they responded they have mobilized various types of technical semantic and syntactic and mixed records algebraic graphical and numerical. This shows that much of students are able to articulate linear functions and trinomial functions with curves.

Concerning the last question, more than 50% (76 among 143), did not give an answer, while students answering these questions have mobilized different techniques.

The syntactic techniques of the algebraic register appeared in 28 answers among 67 and contained no exact answer. The semantical techniques of the graphic register appeared in 18 copies, among them 4 correct answers were found. Two answers mobilized the mixed technique and they were correct.

The other answers that had no connection with the exercise; they were classified in “other types of responses”.

Obtained results, on the three exercises, highlight that pupils use the syntactic techniques of resolution as soon as they are available even if intermediate questions should call for semantics treatments (graphic or numerical). On the other hand, we pointed out that a rather large percentage of pupils and students use the semantic tool of resolution only if it is required by the type of task besides, we observed rather remarkable differences between the procedures of resolution of the exercises of the questionnaire between students of the same level. In this respect, we think that such differences could be linked to differences in the practice of the teachers in class with respect to the dialectics between syntax and semantics.

CONCLUSION

The semantics logical approach, referring to the notion of satisfaction of an open sentence and of contentment has permitted us to identify difficulties that the pupils could face when working with (in)equation and function.

Taking explicitly in account the relationship between equations, curves and functions, enlightens the necessity of semantics that is largely neglected in the Tunisian programs and textbooks.

Articulating the approach we have developed, with the study of effective teaching practices could provide paths for examining in which measure the syntactic or the semantics preferences that we have observed is or is not linked to the teachers’ preferences. Our experimental investigations have pointed out the relevance of using semantics when syntactic techniques have been show to be ineffective. In addition, we have shown that many students fail to take in account the relationship between the equation, curves and function, in line with the fact enlighten by our analysis that it is not clearly made explicit in the textbooks.

Finally, we think that the investigation of the same notions of (in)equation and function in domains others than numerical domains, which appear in advanced mathematics would be promising for the didactic analyses in higher education. We actually investigate the interest of our logical point of view in reference to Predicate calculus for studying in an educational perspective (in)equations in various domains (matrix, vector, functions etc.) Indeed, the notion of open sentences and satisfaction by an element provides a very general frame for the concepts of (in)equation that we suspect to be fruitful.

NOTES

[1] Our translation.

[2]We have chosen in our work not to consider the second block in praxeologies (the logos) due to the fact that in Tunisian program and textbooks, there are very few references to technology and theory.

[3]We mean by type of technique corresponding to a definite type of task.

[4]Level of operating knowledge.

[5]Techniques are expected to mobilize when mixed in the treatment of mathematical objects, both two points of syntactic and semantic.

[6]They were students with high achievement attending « Classes préparatoires aux grandes écoles », that prepare them to the competitive examinations for entry in Engineering Schools.

[7]The different types of techniques articulating different registers are detailed in (Kouki, 2008).

[8]Classes of second year secondary science section, classes of third year secondary mathematics section and a preparatory class specialty technology.

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