HOW DEFICITS IN ARITHMETIC EDUCATION INFLUENCE THE LEARNING OF SYMBOLIC ALGEBRA

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Abstract *In line with a design science project, which is aimed at designing a theory-driven learning environment for teaching symbolic algebra by comparing geometric quantities in early grades, it is investigated how students' previous (arithmetical) knowledge and new learned knowledge about symbolic algebra are intertwined. This paper will present reasons for this non-arithmetical approach to early algebra and will explain why algebra education may still benefit from previous arithmetic knowledge. Furthermore research results are presented that will illustrate and discuss how arithmetical knowledge and patterns of action acquired in previous arithmetic education affect the learning of algebra.*
Keywords: *Early Algebra, Symbolic Algebra, Didactical Contract, Algebra History*

INTRODUCTION

Various international researchers work on questions concerning problems of students' approach to school algebra (see Carraher & Schliemann, 2007). Hence there is detailed knowledge about which difficulties exist and many ideas how to face these difficulties. One idea is to introduce algebra in earlier grades so called early algebra. There are different reasons for following this idea. An important reason for teaching early algebra is that there is evidence for arithmetic, as it is taught in school, affecting negatively children's ability to learn algebra (McNeil, 2004). But there is little research about how exactly arithmetical education interrelates with the learning of algebra.

In the following possible difficulties that result from the different handling of natural numbers and variables will be described. Since the underlying learning environment for the research presented in this paper is based on geometry possible difficulties and benefits of a geometric and an arithmetic approach will be contrasted. For this purpose comparison also is linked to a historic perspective. The paper continues with the methodology of the investigation and presents first results concerning the question what happens when knowledge and patterns of action students acquire in arithmetic and algebra education converge.

THEORETICAL FRAMEWORK

"When we look at school tradition in different countries [...] algebra is generalised arithmetic." (Lins & Kaput, 2004, p.50) In that tradition a lot of research was done and is still going on about number patterns and numerical reasoning in early grades (Carraher & Schliemann, 2007). This happens by conviction that these contents can

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give students a good preparation for their approach to algebra. But to tie algebra to natural numbers provokes multifaceted difficulties.

Algebra as generalised arithmetic?

If students learn about the relations of natural numbers they learn, that they can find the information of these relations in the number symbols. One example: If a student compares **4** and **5**, he does not need the information **4 < 5.** Every student with a little number sense knows that **4** is smaller than **5**. If the student writes down **4**<**5**, she is just recording an information that the numbers **4** and **5** already contain, independent of which sign is written between those numbers. Respectively the student would not accept 4>5 or 4=5 because the sign between the numbers does not represent the true relation between those two numbers. Algebra tasks that are based on natural numbers usually consider merely tasks of elementary number theory with natural numbers and one variable and thereby symbols like **n** or **n+1**. Here too, the student easily can find the relation of the variables by looking at the symbolic expression of the variable, e.g. **n** is always smaller than **n+1**.

Problems arise when algebra is expanded to rational or real numbers and more generalized variables are used. Now the relation of two or more variables is not given by their symbolic expression any more, but by the relation and operation signs between the symbols. If you write **x** and **y** instead of **n** and **n+1** the relation of **x** and **y** is unknown until you know **x < y (**or more exactly **x+1=y**). The student would also accept **x>y** or **x=y** or in case of a case distinction even all of these three relations.

Tasks based on natural numbers and one variable are valuable for training generalised number reasoning and number sense and meaningful especially for young children. But the logical consequence of those tasks is arithmetical number theory. In preparation for elementary algebra it is important to concentrate not on numbers only but on operations and relations that are not bound to specific numbers.

In the following a selection of typical difficulties with algebra are specified (see Carraher & Schliemann, 2007, p. 670):

- equal sign is seen as unidirectional operator, with the task on the left and the result on the right side,
- focus is on finding answers, not on how to find the answer,
- difficulties in recognizing the commutative and distributive properties,
- non-use of mathematical symbols for expressing relationships of quantities;

On closer inspection, one of the reasons for these difficulties may be the focus on natural numbers, especially the number sense concerning these numbers, and the lack of concentration on operations and relations on a more generalised idea of number in the early years of mathematics education.

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These difficulties mostly are not based on a lack of mathematical knowledge, but on using inadequate ways of dealing with mathematical problems, that students acquired in previous mathematics education. In this paper the concept "patterns of action", which will be explained in the methodology section, is used for these ways of dealing with mathematics. It is most likely, that there are more patterns of action that originate from arithmetic education and hinder a good approach to algebra.

There are several ideas to counter the problem caused by these patterns of action. One idea is to change curricula, train teachers and in so doing try to teach arithmetic in a more algebraic way. This is already happening, for example as early algebra. Research literature on early algebra shows the gains from such approaches (see Carraher & Schliemann, 2007). But the changes in primary mathematics classrooms are slow and "mathematics equals calculating" is deep-seated in our society. So there is also a need for research about how to teach students that already have acquired patterns of action which hinder their learning of algebra. The idea of the learning environment this paper based on is to exclude arithmetic when teaching algebra and to bring arithmetic and algebra together later.

Teaching algebra without arithmetic?

If one talks about teaching algebra, one talks about teaching two very different but deeply connected subjects. On the one hand students have to learn algebraic reasoning. This can happen and is practised without using symbolic algebra. On the other hand they have to deal with symbolic algebra, a kind of language which has to be learned and practised. This contains not only the rules that can be conducted by a computer algebra system but also and especially modelling with variables and seeing the meaning behind symbolic expressions. A proper learned algebraic language in turn can serve as a problem solving tool for algebraic thinking. A good approach to algebra requires both.

In traditional school algebra a syntactic way of using symbolic algebra precedes algebraic problem solving by using this symbolic algebra. The consequence is that students focus on the symbols and have problems to give variables a signification (see e.g. Malle 1993). As soon as symbolic algebra comes into play the question arises: How can students productively deal with algebraic reasoning by using algebraic symbols whose signification is also not yet known properly? To give symbolic algebra a meaning it should be connected with a mathematical subject students are already familiar with, like numbers and arithmetic. Independent of the problems displayed above a meaningful handling of natural numbers may foster a meaningful handling of variables. Is it really wise to exclude arithmetic? Or is it possible to find an approach that is not excluding arithmetic compulsorily but rather is compensating the shortcomings of arithmetic education?

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A historic point of view

To learn more about the learning of symbolic algebra an additional look at the development of symbolic algebra from a historic point of view can be helpful. The reason for a look at the history of algebra is not that the ontogenesis of students' understanding of the symbolic algebra has to follow the phylogeny of symbolic algebra (see Harper, 1987). Rather a look at the history of the development of symbolic algebra can help to identify specific problems with and chances for learning to deal with symbolic algebra. Rhetoric and syncopated algebra were used over centuries before symbolic algebra emerged. Hence the introduction of symbolic algebra may also challenge students. The main achievement of symbolic algebra are not letter variables, since letter variables for unknown numbers have been used already in syncopated algebra, but the consequent use of letter variables for known numbers for the description of operations with and relations of these variables.

One may reason that geometric algebra was an obstacle for symbolic algebra because for working with symbolic algebra one has to detach oneself from the graphic power of geometry (Krämer, 1988). On the other hand geometric algebra is supporting a holistic, object-oriented view on algebra (Sfard, 1995). Vice versa the introduction of symbolic algebra allowed a new view on geometry, called analytical geometry.

This historical process cannot be transferred one-to-one to school curriculum because the students do not have an extensive education in geometry before their approach to algebra. But they do have an extensive education in arithmetic and it is possible to draw parallels between the role of geometry for symbolic algebra at that time and the role of arithmetic for symbolic algebra in school (see Figure 1).

Introducing symbolic algebra with...	
Geometry	**Arithmetic**
Advantage	
A graphic basis for working with variables allows a holistic view on the task, the operations and relations.	A numerical basis for working with variables allows a pool of abstract examples for operations and relations.
Disadvantage	
"Graphic power" ties symbolic algebra to concrete geometry.	Properties of numbers hinder the view on operations and relations.
Perspective	
Analytical geometry gives new insight into geometry.	"Algebraic arithmetic" gives new insight in arithmetical operations.

Figure 1: Relation between symbolic algebra and geometry respectively arithmetic

The comparison shows, that the advantages of a geometric approach may cancel out the problems of arithmetic education by shifting the focus from properties of numbers to properties of operations and relations. Later the inclusion of arithmetic into the learning of symbolic algebra may allow an expanded structural basis for symbolic algebra which detaches students' thinking from concrete geometric objects.

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This is the basic idea of the learning environment this paper is based on. For judging the learning environment it has to be revealed which knowledge and patterns of action come into being if arithmetic and geometric-algebraic knowledge converge?

METHODOLOGY

The research that is underlying this paper is based on the paradigm of Design-Based Research which "blends empirical educational research with the theory driven design of learning environments" (The Design-Based Research Collective, 2003, p. 5). The aim of the research project is the development of a theory-driven learning environment for introducing symbolic algebra after primary school based on geometry. After several pre-studies (see Gerhard, 2009) a teaching experiment with 10-11 years old students of a 5th grade of a grammar school was conducted. Starting point of the learning environment is the El'konin-Davydov-Curriculum, which was developed in Russia and refined during teaching experiments by the Measure-Up-Program at the University of Hawaii. For a detailed description of the teaching experiments see Dougherty (2008) and Davydov (1975).

The teaching experiment took part at the end of the school year. During the school year arithmetic lessons contained repeating basic arithmetic operations, place value system, calculating with decimals and prime numbers. Topics of geometry lessons were the coordinate system and basic ideas about circles and angles. The topics length, area and volume were not taught yet but taught after the teaching experiment. So for the teaching experiment had to be considered that students had to use intuitive ideas of area and volume without using multiplication. Figure 2 shows the basic principles of the learning environment and a sample task.

Basic principles	Sample task
• Comparing geometric quantities of unknown size • Calculation with letters that characterise unknown sizes of geometric quantities • Visualisation with auxiliary drawings. • Analytical idea: choosing letter variables for unknown values • Finding as many relations as possible. • Write down relations as equations and inequations. • Use basic transformation rules (Malle 1993, e.g. a+b=c ⇔ b = c-a ⇔ a = c-b)	A piece of land has area p. The piece of land contains of two parts. One part is grassland, the other part is farmland. The grassland has area g. How big is the area of the farmland? 1) Make a drawing which should be as simple as possible. 2) Write down all equations and inequations you can think of the diagram. 3) Which equations and in-equations tell you, how big the farmland is?

Figure 2: Learning Environment

The research focus is on the question, how new knowledge about symbolic algebra taught in the teaching experiments might interact with old knowledge achieved in previous arithmetic education. The approach to this question is explorative and therefore hypothesis generating. For allowing the investigation of a long term effect

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additional problem-centred, semi-standardised interviews were conducted 6-7 month after the teaching experiment. During the interviews students were confronted with arithmetic story problems that were modified by using letters instead of numbers. The transcripts of the interviews together with the students' written products were analysed using an analysis tool which was designed for this purpose (see Figure 3).

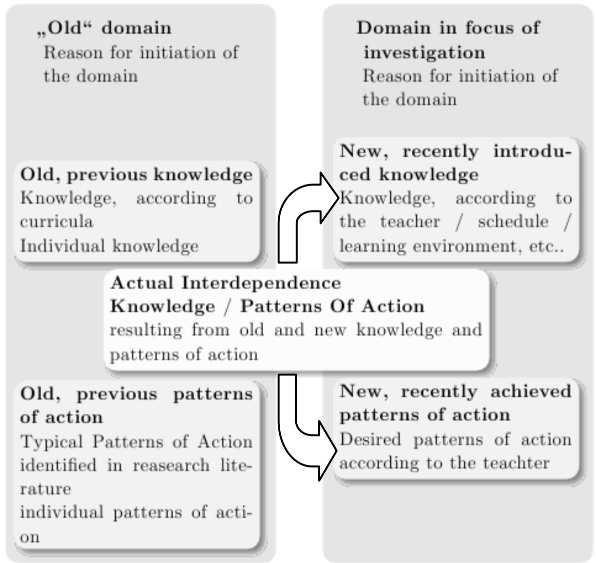
Sample interview problem	Interdependence Analysis Tool (revised 2012)
Running for a good cause Mrs. Gunther takes part in a run in stages. The more kilometres she runs the more donations are paid to the organizer. The run lasts from Monday to Saturday. From Monday she runs a hours per day. On Saturday she only runs half of the time. If she makes b kilometres per hour, how many kilometres does she run altogether? (Structure of the task: $5\cdot a\cdot b + (a:2)\cdot b = L$)	

Figure 3: Interview design and revised interdependence analysis tool

The concept "pattern of action" is based on Brousseau's didactical contract (see. Hersant, M. & Perrin-Glorian 2005). "Situating a problem within a certain mathematical field guarantees that certain techniques will appear natural and will be favoured whereas others will be improbable." (Hersant, M. & Perrin-Glorian 2005, p.118). Patterns of actions are defined as this certain techniques which are bound to the **domain** in which a student believes to be. As two domains, one arithmetic, the other geometric-algebraic, are confronted, the patterns of actions concerning both domains have to be considered. Patterns of actions depend on the **didactic status of knowledge**. Usually patterns of actions connected with old knowledge are deeper-rooted than patterns of actions connected with new recently taught knowledge.

The analysis tool is used as follows: Starting point is an interpretation concerning the reasons for how the student acts in the actual situation. These reasons are classified according to domain, knowledge and patterns of action, old as well as new. The analysis starts with expected patterns of actions and knowledge in both domains and ends with the students' supposable actual algebraic patterns of action and knowledge influenced by previous arithmetic patterns of action knowledge.

The performance of two students on the sample task is used to illustrate how previous patterns of actions and knowledge influence the students' knowledge about and patterns of action used on symbolic algebra. For illustrating the findings a low and a high achieving student in arithmetic were chosen. The analysis is part of case

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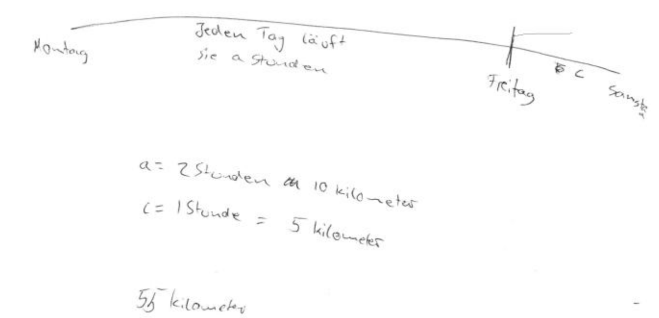
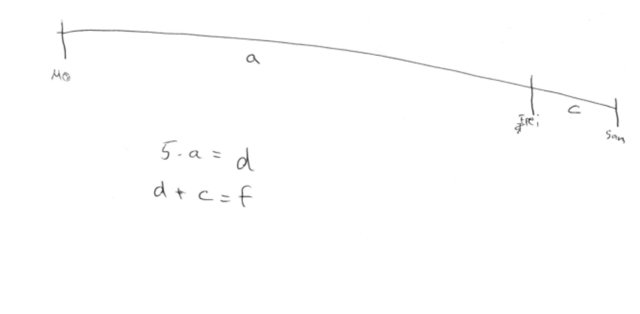
studies with twelve students, which differ in performance on arithmetic and algebra. With these case studies relationships of old and new knowledge and patterns of actions that are consistently used by several students and therefore qualify for generality will be identified. The analysis of the case studies will be finished in December 2012, so it is possible to present more general results at CERME 8.

Dorian, an outstanding student, reads the task and started immediately to write down and explain a solution of the task (see Figure 4), using letters and numbers.

1	Hm- well, than is that here, then she runs Monday to Friday, that are 5 days every day **a** hours,
2	that means **a** plus **a** plus **a** plus **a** plus **a-** (writes ‚a+a+a+a+a‘) and then she runs only half as many and
3	that means she also runs only half as many, that means again once more plus **a** divided by 2 (writes
4	‘+a:2‘) [...] And that is then- only the time (writes ‚=T‘). If we now, if we, if now **a** hours are 2 hours,
5	then she would walk there always 2. (points to the first **a**, writes ‘2+2+2+2+2’) and here always
6	1 (writes ‘+1’) and that would be then two four six eight ten (writes ‘=11’) eleven hours. And then
7	she makes 3 Kilometres per hour, well in 2 hours then 6 plus 6 plus 6 plus 6 plus 6 plus 3 (writes
8	‘6+6+6+6+6+3’), that would be one two three four five- thirty three (writes ‘=33’). And that is so a
9	while, that is the same as this (points one after another at the first and second written number
10	equation). Then I could her, you already see, that looks nearly the same, **b** plus **b** plus **b** plus **b** plus **b**
11	plus **b** divided by 2 (writes ‘b+b+b+b+b+b:2’) and that would be kilometres (writes ‘=K’).

Figure 4: Dorian solving the task “Running for a good cause”

Christina (see Figure 5), a low achieving student, begins with an auxiliary drawing. Then she is solving the task with invented numbers while at the same time working on the letters (line 21-25). After that she is solving the task with letters (line 31-32).

Solution with numbers	Solution with letters
	

21	And the, erm, has to be from Monday till Friday, erm, every day, erm, <u>**a**</u> hours. Well, you could maybe
22	**a**- **a** equals 2 (writes ‘**a**=2 Stunden’ (***a***= *2 hours*). [...] well write down. Every day she (writes ‘Jeden
23	Tag läuft sie **a** Stunden’ *(Every day she runs **a** hours)* under the line) runs **a** hours. And then, if she
24	from, erm, on Saturday runs only half of the time, erm, this has to be so 1 hour. Well here, if you, if
25	you (incomprehensible) **b**, no (writes **b** under the line and cancels it out) **c**, **c** is the one hour?
...	
31	Well, for my part 5 times **a**? [...] 5 times **a** equals- Then you shall invent letters for the result? [...]
32	5 times **a** equals **d**, for my part- and yes then **d** plus **c** equals for my part **f**.

Figure 5: Christina solving the task “Running for a good cause” with illustrating notes

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RESULTS

Dorian's and Christina's examples illustrate the following patterns of action: **Keyword-Strategy**, **Inventing-Letters-Strategy,** **Inventing-Numbers-Strategy**, **Letters-As-Quantities** and **Use of Auxiliary-Drawings.**

Keyword-Strategy

Using keywords to find out which operation to conduct is an old *pattern of action* rooted in previous mathematics education. For example the keyword "remove" leads to subtraction. Because keywords will not change when letters are used this strategy still work with letters. Dorian may have chosen the operation ":2" because of the keyword "half" (line 2-4). Keywords have a positive effect, if the calculation strategies they invoke are explicit available. Christina was not able to express "half" by ":2". Instead she wrote **c** (line 25) and defined, that **c** is half of **a** (line 24). She is not doing this explicit but via **Inventing-Number-Strategy** (see later). This is a sign of a lack in *old arithmetic knowledge*, an **insufficient operational understanding**

The Keyword-Strategy is adopted by the students for the use in the *new algebraic domain*, but they will only have a positive effect, if the calculation strategies they invoke are compatible with the corresponding symbolic description. In another task the students had to find out how often a distance **x** fits into a distance **y**. Transcripts of several students working on that task show that another keyword that cannot easily be expressed in symbolic language is "how often fits in". The students learned in previous arithmetic education the *old pattern of action* that if **x** and **y** are natural numbers and **y** is a multiple of **x** it is possible to calculate the solution **L** with the help of repeated addition or multiplication, instead of division. But repeated addition without knowing how many addends there are is difficult to express with symbolic algebra. The corresponding symbolic description for multiplication is: **L·x=y**. As the solution **L** of the problem is not identical with the "result" **y** of the equation, this equation is not compatible with **Process-Orientation (**see later), an *old pattern of action* which draws the student's attention to results instead of relations.

Inventing-Letters-Strategy

Inventing letters results directly from a fundamental algebraic idea, the **Analytical Idea**, which was taught as *new algebraic knowledge* as part of our algebraic learning environment. It results in the *new pattern of action*, that if a value is unknown you can write a letter that can be used like a number. Thus one is able to work with the unknown value like an actual number assumed this *new algebraic knowledge* exists. Dorian is combining the analytical idea with an *old pattern of action*, **Process-Orientation**. He is allowed to use letters for unknown values and the result of his letter calculations (line 2-4 and 11) is unknown. So he concludes that he is allowed to invent letters for results, **T** for time and **K** for kilometres. Of course an algebra teacher would welcome if Dorian accepts the expressions **a+a+a+a+a+a:2** and

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b+b+b+b+b:2 as result. But these expressions have a lack of closure and students who are tied to process-orientation will hardly accept this. However, inventing a letter as result may be a good *new pattern of action* for a start.

Christina also uses this strategy as *new pattern of action* (line 25 and 31). At first she uses **b** for half of the time but realised properly that **b** is already given away (line 24-25). Instead she uses **c**. The difficulty with inventing letters for unknown numbers is that students have to relate every new letters symbolically with the variables already in use. Christina is not able to relate **c** with **a** symbolically, most likely because of **insufficient operational understanding**. However if she gains more experience with the use of variables this may help her to identify insufficient operational understanding and make explicit her *implicit old knowledge* about operations. Here may her *old arithmetical knowledge* benefit from *new algebraic knowledge*.

Inventing-Numbers-Strategy

Dorian replaces letters by invented numbers, conducts the calculations and translates the conducted calculations back in letter expressions. Later Dorian will say that he has used the numbers as aid for thinking.

Old knowledge about **setting up and manipulating number expressions** as well as **number operations** can support *new knowledge* about **setting up and manipulating letter expressions** as well as **letter operations.** But *patterns of action* like choosing inappropriate numbers, replacing the wrong letters or inventing numbers for unknown values without keeping in mind the relations to other values, which are deeply connected with *old knowledge* may cause problems.

Christina is inventing the numbers for the hours per day and hours for half of the time considering the relation between the variables (line 22-24). But she does not explain that **c** is half of **a**. Instead she uses the numbers to make this relation explicit by explaining that **a** equals 2 hours, 1 is half this time and **c** equals 1 hours. Probably the *pattern of action* of choosing easy numbers is inappropriate to put in her mind that the half can be calculated explicitly by :2.

Letters-as-Quantities

Dorian was taught the *new knowledge* **letters signify unknown numerical values of quantities**. Earlier in the interview he explained that letters stand for numerical values, but he consequently refers to the quantities if he is talking about the value of the quantities. Therefore he is using Letter-as-Quantities as *new pattern of action.* He is calling **T** time (line 4) and **K** kilometres (line 12). At first this seemed to be a metonymy because it is easier to talk about kilometres instead of the number of kilometres. Especially in geometry this double meaning is widely accepted, e.g. talking about the side **s** if actually meaning the length **s** of the side. But using multiple meanings as *old geometric pattern of action* may cause problems.

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At the end of the task (line 10-11) Dorian is using **b** for both kilometres per hour and kilometres per day, but not for all kind of kilometres as the kilometres in total are signified as **K**. One reason may be that he is because of using the **Inventing-Number-Strategy** and as a result calculating the kilometres per day with mental arithmetic, for him the relation between hours and kilometres does not become explicit. Thus he may not realize that the status of the kilometres has changed. Christina does not clarify the relation between hours and kilometres, too, and mixes up the letters that signify the quantities. Again *new algebraic knowledge* may be a chance, here for making explicit the students' *implicit old knowledge* about relations

Using auxiliary drawings

Christina had problems to begin with solving the task. Only after she reassured that she can use "lines" she started working. While she was working she comprised the drawing many times (line 22-25). In the end she translates the drawing into equations (line 31-32). This emphasises, that using auxiliary drawings is an important n*ew pattern of action* for her. Dorian resigns to use a drawing but he later refers to the 'graphic power' of the equations by stating "you already see, that looks nearly the same" (line 10). He seems to be able to 'see' the explicitly relations in equations without using drawings. Dorian's new algebraic knowledge benefits from this *old knowledge*. But unfortunately this *old knowledge* is not self-evident for students.

The problem with auxiliary drawings is that students can only see in the drawings what they have put inside. Christina's drawing does not contain enough information about the relation between hours and kilometres and hence she is not able to gain equations that represent the relation. Insufficient drawings can be a result of the *old* pattern of action of concentrating on numbers instead of relations, which again is a result of **Process-Orientation** and highly supported by **Keyword-Strategy.**

DISCUSSION

The influence of previous arithmetical education on the learning of algebra is complex. The examples of Christina and Dorian reveal that in particular the lack of explicit knowledge about operations and relations hinder a good approach to algebra. Patterns of action and knowledge about proportions of numbers, both achieved in previous arithmetic education account for clouding this knowledge. Giving algebra a strong geometric or more precisely graphic basis may help to make operations and relations explicit. However, whatever approach to algebra is chosen in later years, it will not be successful if it does not consider the multifaceted influence of previous education on the learning of algebra.

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