

THE CONSISTENCY OF STUDENTS' ERROR PATTERNS IN SOLVING COMPUTATIONAL PROBLEMS WITH FRACTIONS

Gerald Wittmann

Pädagogische Hochschule Freiburg

Error patterns in solving computational problems with fractions are well known. However, the question whether these patterns are consistent or not is still unanswered. This paper presents the design and results of an empirical study focussing on this question in the case of computational problems with fractions. It shows that a considerable part of error patterns and furthermore of approaches is not consistent. This implies that the concerning students do not have any strategies when dealing with computational problems. Instead, their solutions are not rationally chosen but emerge while treating the problem.

1 ERROR PATTERNS AND THE CONSISTENCY OF ERROR PATTERNS

Learning fractions in secondary mathematics education has two aspects: The students should build up conceptual knowledge (e.g. the interpretation of a fraction as a part of a whole) as well as procedural knowledge (e.g. computational skills like adding or multiplying two fractions). This paper focuses on the second aspect. It investigates the consistency of students' error patterns in solving computational problems with fractions.

In that regard, a *computational problem* is a problem that can be solved by using procedural knowledge only as there are specific algorithms to solve the problem (e.g. adding two fractions by converting to a common denominator). Therefore, neither conceptual knowledge nor heuristic strategies are necessary. Beyond that, a computational problem can be solved in another manner as well: By recall (e.g. knowing that $\frac{1}{2} + \frac{1}{4}$), by “seeing” the result without any calculation (e.g. $6\frac{5}{7}$ as result of $6 + \frac{5}{7}$) or by using conceptual knowledge (e.g. realizing $\frac{2}{9}$ as half of $\frac{4}{9}$ and thus $\frac{2}{9}$ as result of $\frac{4}{9} \cdot \frac{1}{2}$). However, the relationship between conceptual and procedural knowledge and their interdependency still needs further research (Hallet, Nunes & Bryant, 2010).

The *research on students' errors* at computational problems with fractions has a long tradition. Frequently appearing errors are well-known and comprehensively documented in many years (Carpenter, Fennema & Romberg, 1993; Kerslake, 1986; Eichelmann, Narciss, Schnaubert & Melis, 2012). An *error pattern* can be identified when identically structured errors appear in the solution of two or more identically structured problems (Prediger & Wittmann, 2009).

The error patterns relevant for this study are given in table 1. Nearly all of them can be characterised as a combination of well-known steps of solution processes being

inappropriate for the underlying problem. A common model to explain this fact is the “bug-repair-theory” (Brown & VanLehn, 1980).

Multiplication of two fractions with a common denominator: $\frac{a}{b} \cdot \frac{c}{b} = \frac{a \cdot c}{b}$

Addition or subtraction of two fractions: $\frac{a}{b} \pm \frac{c}{d} = \frac{a \pm c}{b \pm d}$

Addition of a fraction and an integer: $\frac{a}{b} + c = \frac{a+c}{b}$, $\frac{a}{b} + c = \frac{a}{b+c}$, $\frac{a}{b} + c = \frac{a+c}{b+c}$

Table 1: Main error patterns

An error pattern is called *consistent* referred to a student if the student deals with two or more identically structured problems within a short period of time and produces the same error pattern in each (or nearly each) of the underlying solution processes. For example, if a student deals with several computational problems of adding fractions within a test, the common error pattern ‘adding the numerators and adding the denominators’ may occur in all of his or her solutions – in this case it is consistent. But it may occur that only some solutions show the error pattern and the other solutions are correct – then the error pattern is not consistent. Therefore, the consistency of an error pattern has to be investigated at an individual level.

The question *whether error patterns concerning computational problems with fractions are consistent or not* still lacks research. In literature there is little evidence. Earlier studies always consider the entire test population as they aim primarily at the identification of error patterns or the frequency of their occurrence (e.g. Hart, 1981; Padberg, 1986). Solely Padberg (1986) differs between ‘typical errors’ (error patterns occurring frequently in the whole population) and ‘systematic errors’ (error patterns consistently occurring at an individual level). However, he does not provide any data answering the question if error patterns are consistent.

Recent studies concerning computational problems with fractions show a broad range of (correct or incorrect) individual solutions (Hennecke, 1999, using methods of graph theory) or demonstrate that students’ procedural knowledge about fractions falls apart into several incoherent classes of problems, even adding and subtracting fractions (Herden & Pallack, 2000, by means of cluster and factor analyses). Beyond that, studies on solutions of linear equations in algebra lead to the hypothesis that error patterns are often not consistent (Tietze, 1988; Stahl, 2000).

Regarding the question whether error patterns are consistent or not, a special view on students’ solutions is very helpful. This study gathers students’ *approaches* to computational problems with fractions. An approach is a general form of solving the problem, comprising many individual solutions. Taking the addition of fractions as an example, two approaches can be distinguished: first converting both fractions to a common denominator, second adding the numerators and the denominators separately.

Though the presented study can be seen in the long tradition of research on students' errors, the research question is formulated more generally, regarding not only errors but also correct solutions: *Are approaches for computational problems with fractions consistent?* The answer to this question can be seen as one step to understand and model the way in which students solve computational problems.

2 DESIGN OF THE EMPIRICAL STUDY

In order to examine the consistency of error patterns empirically four sets of six problems were developed (table 2).

Multiplication of two fractions:	$\frac{4}{9} \cdot \frac{1}{2}$	$\frac{8}{5} \cdot \frac{3}{7}$	$\frac{4}{5} \cdot \frac{3}{5}$	$\frac{5}{13} \cdot \frac{3}{13}$	$\frac{5}{2} \cdot \frac{5}{2}$	$\frac{4}{15} \cdot \frac{4}{15}$
Addition of two fractions:	$\frac{1}{2} + \frac{1}{4}$	$\frac{2}{7} + \frac{4}{5}$	$\frac{2}{13} + \frac{4}{9}$	$\frac{1}{9} + \frac{1}{8}$	$\frac{2}{5} + \frac{2}{5}$	$\frac{6}{11} + \frac{6}{11}$
Subtraction of two fractions:	$\frac{3}{4} - \frac{1}{2}$	$\frac{6}{7} - \frac{2}{5}$	$\frac{7}{10} - \frac{8}{15}$	$\frac{2}{3} - \frac{2}{7}$	$\frac{13}{8} - \frac{11}{8}$	$\frac{7}{13} - \frac{7}{13}$
Addition of a fraction and an integer:	$\frac{1}{4} + 1$	$\frac{2}{3} + 7$	$\frac{2}{13} + 6$	$4 + \frac{2}{3}$	$6 + \frac{5}{7}$	$7 + \frac{1}{13}$

Table 2: Four sets of computational problems

The problems within a set differ from each other in the given numbers. For example, the set of problems for multiplication of two fractions contains three pairs of problems: (1) with different denominators, (2) with the same denominator but different numerators and (3) with equal fractions. Each of the pairs contains a problem with large numbers and a problem with small numbers.

Each test sheet comprises 18 tasks in three of the four sets of problems which are arranged in nine variants in a random order to avoid serial effects. Thus there are 36 different test sheets. Three more problems without any relevance were inserted to scatter a series of problems of one set caused by the random order.

The students' working time was 40 min and they were allowed to solve the problems in a free way, e.g. calculate mentally or in a written form but not with a calculator. 428 students participated in the test in July 2011. As the students worked only on three of the four sets of problems due to the construction of the test, the number of solutions to each problem is lower. The students were 6th- and 7th-graders of Real- and Werkrealschule in Baden-Württemberg, a federal state in south-west Germany. A considerable number of low-achievers attend these schools.

Unlike traditional research on students' errors, the translation of the data into codes focuses on approaches rather than on correct or incorrect solutions. For this reason, simple computational errors are ignored, e.g. errors concerning the multiplication tables like $7 \cdot 9 = 61$. In order to answer the question whether an error pattern is consistent or not, these errors do not matter because the research interest focuses on a correct *approach* and not on a correct or incorrect solution. This way of transforming the data reduces the data to a necessary extent and on the other hand features the

required precision. However, in some cases the transformation of the data is an interpretation of written calculations and hence vulnerable to artefacts. A crucial prerequisite is that different approaches can be separated clearly. In the presented study concerning fractions this succeeds very well. Table 3 shows the coding manual taking the example of multiplication.

0	Missing.
1	Correct approach (whether calculated mentally or in a written form; the solution itself may be correct or incorrect).
3	Main error pattern ‘denominator maintained’ (in the case of fractions with different denominators after converting to a common denominator); further errors might be encountered, e.g. errors concerning the multiplication tables.
9	Other (rare error patterns e.g. multiplication with the reciprocal or performing an operation or incomprehensible solutions).

Table 3: Coding manual taking the example of multiplication

Afterwards the data is analysed with methods of descriptive statistics: frequency tables, cross tables (supplemented by a significance test) and tables of ordered tuples of codes.

3 SELECTED RESULTS

Some results of three fields are given: multiplication of two fractions, addition of two fractions and addition of a fraction and an integer. Thereby the process of data analysis is illustrated.

3.1 Multiplication of two fractions

Table 4 shows the frequencies of the approaches concerning the six multiplication problems (N = 315).

Coding	$\frac{4}{9} \cdot \frac{1}{2}$	$\frac{8}{5} \cdot \frac{3}{7}$	$\frac{4}{5} \cdot \frac{3}{5}$	$\frac{5}{13} \cdot \frac{3}{13}$	$\frac{5}{2} \cdot \frac{5}{2}$	$\frac{4}{15} \cdot \frac{4}{15}$
0 Missing	24	31	14	21	13	21
1 Correct approach	226	228	199	167	207	176
3 ‘Denominator maintained’	30	25	74	102	53	91
9 Others	35	31	28	25	42	27

Table 4: Multiplication of two fractions – Frequency of approaches

A correct approach occurs more often in problems with different denominators than in problems with a common denominator while the main error pattern ‘denominator maintained’ pertains the opposite. Besides, some students act consistently in an extreme way: They convert fractions to a common denominator and subsequently

deal with the multiplication according to the main error pattern ‘denominator maintained’. So do 30 students at the first problem and 25 at the second (among them 17 at both problems).

Considering the problems with a common denominator, the main error pattern is in each of the two pairs of problems obviously more frequent when the denominator is larger. Table 5 shows a detailed analysis of this effect in the case of multiplying two equal fractions: 235 of 315 students solve both tasks by the same approach while 80 generate different approaches. In particular, 31 students performed a correct approach with the denominator 2 (‘small denominator’) while they maintained the denominator 15 (‘large denominator’). The McNemar-Bowker-test confirms that the asymmetry of the cross table is significant ($\chi^2 = 39,015$; $df = 5$; $\alpha < 0,001$). This test checks against the null hypothesis namely that the cross table is symmetrical along the main diagonal.

		$\frac{4}{15} \cdot \frac{4}{15}$ (‘large denominators’)				
		0	1	3	9	
$\frac{5}{2} \cdot \frac{5}{2}$	0 (‘small denominators’)	11	1	0	1	13
	1	7	159	31	10	207
	3	0	2	50	1	53
	9	3	14	10	15	42
		21	176	91	27	315

Table 5: Multiplication of two equal fractions – cross table

Table 6 focuses on the approaches to the four multiplication problems with a common denominator that can be estimated as identically structured.

	So many students performed x -times ...				
	0	1	2	3	4
Correct approach	79	28	38	35	135
‘denominator maintained’	199	20	28	28	40

Table 6: Multiplication of two fractions with the same denominator – frequency of approaches

Multiplying two fractions with a common denominator, 135 of 315 students treated all four problems according to a correct approach, while e.g. 38 did so at only two out of four problems and behaved differently at the other two problems. 199 students never followed the approach ‘denominator maintained’ and 40 students used it each

time while e.g. 20 students generated this error pattern only at one out of four problems and otherwise dealt with the remaining three problems.

In addition to table 6 some more interesting facts can be given: 11 out of 40 students who maintained the denominator at all four problems worked on the two problems with different denominators in the same way, after converting the denominators. Therefore, the error pattern is consistent at all six multiplication problems. A further 11 students who maintained the denominator at all four problems solved the two problems with different denominators according to the correct approach – the approaches are consistent only within the cases of common or different denominators.

3.2 Addition of two fractions

Regarding the addition of two fractions, the frequency of the main error pattern ‘separate addition of the numerators and the denominators’ varies little across all six problems, as table 7 shows (N = 347).

Category	$\frac{1}{2} + \frac{1}{4}$	$\frac{2}{7} + \frac{4}{5}$	$\frac{2}{13} + \frac{4}{9}$	$\frac{1}{9} + \frac{1}{8}$	$\frac{2}{5} + \frac{2}{5}$	$\frac{6}{11} + \frac{6}{11}$
0 Missing	19	41	116	62	16	16
1 Correct approach	227	205	116	167	224	230
3 Main error pattern	73	79	84	78	92	78
9 Others	28	22	31	40	15	23

Table 7: Addition of two fractions – frequency of the approaches

However, another effect is striking: the larger the two denominators are, the less often the right approach occurs and the less often students work on the problem but skip it, with a range of 19 to 116 students. This can be interpreted as follows: to leave out a problem is a reaction to ‘large’ denominators when the finding of a common denominator affords a higher level of mental calculation. This reaction gives a hint that the regarding students neither have the result of the underlying multiplication problem as a retrievable fact nor can expand it briefly. It is not possible to decide if the students cannot find a common denominator or if they do not want to work on it, perhaps due to a lack of motivation or effort or a low level of frustration tolerance.

Table 8 shows the students’ approaches across the six addition problems as ordered 6-tuples. For example, 110111 in the second line means that the first two and the last three problems are solved correctly whereas the third problem was not done at all. Among 347 students there are 126 different tuples: 18 of them can be found at least three times (these are shown in table 8), another 13 tuples occur only twice and 95 only once.

6-Tupel	Frequency	Percentage	Cumulative percentage
111111	83	23,9	23,9
110111	34	9,8	33,7
333333	32	9,2	42,9
110011	21	6,1	49,0
111133	7	2,0	51,0
333311	7	2,0	53,0
000000	6	1,7	54,8
110131	6	1,7	56,5
333331	5	1,4	57,9
111131	4	1,2	59,1
100011	3	0,9	59,9
110113	3	0,9	60,8
111311	3	0,9	61,7
113333	3	0,9	62,5
119911	3	0,9	63,4
199911	3	0,9	64,3
333313	3	0,9	65,1
999911	3	0,9	66,0

Table 8: Addition of two fractions – processing of six problems

The tuples 111111 and 333333 indicate consistent approaches: 83 students always have a right approach and 32 students show the main error pattern ‘separate addition of numerator and denominator’ in all six tasks. This means that a third of the approaches can be explained by the assumption of consistent approaches whereas two-thirds of the approaches are seen as non-consistent.

Amongst these two-thirds there are two very interesting tuples: Both 111133 and 333311 occur seven times. The first characterises the students who approach the four additions of fractions with two different denominators in a correct way and the two additions of fractions with common denominators according to the main error pattern. The second describes the reverse of the phenomenon. In other words, there are students who ‘fall into’ the error pattern in case of common denominators but there are also students who ‘fall into’ the error pattern in case of different denominators.

3.3 Addition of a fraction and an integer

This section briefly describes two interesting results without showing the full data.

Concerning the addition of a fraction and an integer a more detailed translation of data into code is appropriate (even though there are naturally some borderline cases): First, a mental solution is estimated as a category of its own because it can refer to conceptual knowledge. In fact, about a quarter to a third of the problems is solved mentally. Regarding the approaches, no differences can be seen between problems of the structure ‘fraction plus integer’ and ‘integer plus fraction’ (table 2). The hypothesis that in the case of ‘integer plus fraction’ the solution is ‘seen’ more often and on the other hand in the case of ‘fraction plus integer’ the students more often convert the integer to a fraction and add the two fractions afterwards (c. f. Padberg, 1986) cannot be approved. On the contrary, the problem $\frac{1}{4} + 1$ is solved slightly more frequently mentally than the other five. In consequence, the fact whether the denominator is ‘large’ or not has more influence on the solution process than the question whether the problem is of the form ‘integer plus fraction’ or of the form ‘fraction plus integer’.

Second, the addition of a fraction and an integer does not show only one main error pattern but several error patterns (table 1) which all occur in the test. The approaches of 28 out of 313 students show unambiguously at least two different error patterns. In conclusion, these students do not possess retrievable stable algorithms but only some fragments of algorithms which they apply and combine haphazardly. Furthermore, they lack conceptual knowledge to control their results or the used algorithms.

4 DISCUSSION

The study shows that the numbers given in computational problems with fractions (especially as denominators) have an effect on (1) whether students work on the problem or skip it and (2) what kind of approaches occur. Especially the last finding is astonishing. It was clear in advance that the percentage of correct *solutions* decreases when the denominators get larger as more errors occur e.g. in the multiplication tables. But the fact that the given numbers have an effect even on the *approach* could not have been expected. However, there is the hypothesis that not only and not primarily the size of the numbers is relevant but the question if necessary multiplication tables are known by heart or can be calculated mentally without any difficulties. This hypothesis is an aim of further research.

With regard to the consistency of approaches *four groups of students* can be distinguished for each of the four operations. *Group 1* includes the students who solve all problems of one operation according to a correct approach. About a fourth to a third of the tested population belong to this group. Within this group, two subgroups can be found: students who process the problems in a very flexible way, adequate to the given numbers (e.g. if the numbers are small, the solution is ‘seen’ or

solved mentally), and students with an extreme commitment to a certain approach regardless whether it is appropriate for the posed problem or not ('always convert the denominators before adding two fractions'). *Group 2* contains students whose solutions show the main error pattern without exception. These students may have internalized an incorrect procedure while performing automation exercises. This group is quite small.

The first two groups can be distinguished clearly. Together they comprise those students whose approaches are absolutely consistent. Two other groups include those students whose approaches are not consistent. These two groups cannot be separated clearly but merge into each other.

Group 3 contains the students who show mainly correct approaches but occasionally proceed otherwise. Due to this small amount of different approaches this group is different to group 1. One possible explanation may be careless mistakes that are possibly fostered by the intuitive form of error patterns, by the random arrangement of problems in the test sheet and by the given numbers when they 'fit' into an error pattern.

Group 4 accumulates the students whose approaches are not consistent at all. Even within one set of problems there are two or more different approaches. Presumably, the influence of given numbers is dominant because no stable methods are available. Therefore, special cases (e.g. fractions with a common denominator) are treated differently than general problems. The approaches of group 4 can be interpreted as 'calculating with digits' and as a widely unconsidered processing of numbers given in the problem according to known elements of algorithms. The concerning students do not have any strategies when dealing with computational problems.

As a result, students' work on computational problems quite often appears as a barely controlled and only partly aware process. The approaches are in many cases rather rationally chosen but emergent over time while treating the problem (according to Threlfall, 2009), despite the fact that all of the students have attended a systematic course in fractions.

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