REPRESENTATIONS AND REASONING STRATEGIES OF GRADE 3 STUDENTS IN PROBLEM SOLVING

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In this communication we analyse the representations used by grade 3 students and how they relate to mathematics reasoning on a mathematical problem. Data were collected through audio and video recording in the classroom as well as students' written productions. Students that use schematic representations are those who have more success in solving the problem, in contrast with those that use pictorial and symbolic representations. The reasoning of some students shows interesting instances of making generalizations and working in a systematic way.

INTRODUCTION

The use of mathematical representations plays a fundamental role in students' learning of this subject, which is underlined by curriculum documents in many countries (e.g., NCTM, 2000). Reasoning, another important mathematics process, is highly dependent on the use of representations (Ponte, Mata-Pereira & Henriques, 2012). However, how students use these processes is an issue under-researched, in particular at elementary school level. This communication analyses the representations used by grade 3 students and how they relate to mathematics reasoning when they work on a mathematical problem.

MATHEMATICAL REPRESENTATIONS AND REASONING

In their mathematical work, students use a variety of external representations. Webb, Boswinkel and Dekker (2008) mention three key kinds of representations: informal (produced by students themselves, closely related to the context), preformal (still connected to the context, but including some abstract and formal aspects) and formal (with mathematical notations and language). In their perspective, students begin by using informal representations and gradually move on towards formalization. Some investigations describe the difficulties of students with specific representations. For example, Deizman and English (2001) refer the difficulties that students have in using and understanding diagrams, an important visual representation. The authors consider that these difficulties are related to misunderstanding the notion of diagram (not knowing the structure of the representation and overvaluing superficial characteristics), to incapacity to elaborate adequate diagrams (choosing the diagram that represents the situation or being unable to structure in a diagram the information provided) and to incapacity of reasoning in an proper way with diagrams (making valid inferences regarding the problem).

Reasoning is the process of making proper inferences from given information (Ponte, Mata-Pereira & Henriques, 2012). Mathematics reasoning is thought of as essentially deductive, deriving statements in a logical way from given propositions, but it also may be seen as having an experimental side, making conjectures from specific cases and testing them (Pólya, 1990). Induction is the process of making a generalization by abstracting common features in a number of cases. Abduction is the process of establishing a general hypothesis that may explain a phenomenon from which only some elements are known. Rivera and Becker (2009) present in the following way the relations between abduction, induction and deduction:

> Abductive reasoning involves forming a reasonable hypothesis about the phenomenon. To form that hypothesis, we verify and test the abduced hypothesis several times to see whether it makes sense. When doctors perform an assessment or when jurists analyse a case with incomplete data, the preparatory stage in developing an inference involves abduction (…). Following Peirce, abduction as a concept exists and is used only in relation to induction. Thus, a complete pattern generalization involves complementary acts of abduction and induction and, of course, justification. (p. 217)

According to Lannin, Ellis and Elliot (2011) mathematics reasoning involves processes such as conjecturing, generalizing, investigating why and developing and evaluating arguments. From these, we give especial attention to generalizing (the key process of inductive and abductive reasoning) and justifying (the key process of deductive reasoning). Teachers may understand their students' reasoning from the representations that they use, since these yield a record of their efforts to grapple with mathematics (NCTM, 2000). Mathematical reasoning takes place in solving mathematics problems. As Wickelgren (1974) indicates, solving a problem may be regarded as achieving a goal using certain operations on certain givens. In any problem, besides the explicit information provided, there is always the need to use also implicit information regarding goals, givens and operations. Solving a problem is making inferences (or reasoning) from the available explicit and implicit information in order to reach the stated goal. The global approach to solving a problem is a reasoning strategy. As this author indicates, random trial and error is the "first thing that most people do when confronted with a problem" (p. 46). A most desirable strategy to solve problems is systematic trial and error, that is a "method that automatically [produce] a mutually exclusive and exhaustive listing of all sequences of actions up to some maximum length" (pp. 46-47). As this author points out, between random and completely systematic trial and error "there can be different degrees of systematicness" (p. 47)

RESEARCH METHODOLOGY

This study follows a qualitative approach and was undertaken in a school near Lisbon with the first author as a non-participant observer (Bogdan & Biklen, 1994). The participants are the teacher Fernanda and her 19 grade 3 students (teacher and student names are pseudonyms) who have been together since grade 1. The teacher has been working in this school for the last 10 years and has tenure. She indicated that the students were used to solve problems similar to the one reported on this paper. The class observation lasted for 80 minutes. We also collected students' written productions.

We made content analysis (Bardin, 1977, Laville & Dionne, 1999) of the students' productions and of audio and video recorded interaction among students and between teachers and students. In the content analysis, we followed the three steps defined by Bardin (1977): Pre-analysis, material exploration and processing of results. In the first phase, we defined categories through an open model (Laville & Dionne, 1999), in the second phase, we defined the registration units by subject (Bardin, 1977) and, in the third phase, we made a qualitative approach of iterative construction of an explanation (Laville & Dionne, 1999). We chose this approach because we wanted to develop, through the data, step by step, a logical explanation of the phenomenon observed by examining the existing sense of unit, the interrelationships between these units and the categories in which they are assembled. In analysing students' representations we noticed two main categories (pictorial and schematic) which we further subdivided into finer categories (pictorial detailed/simplified, scheme with dashes/letters). In pictorial representations students made drawings to depict figurative images of the animals; these drawings could be detailed (showing the whole animal) or partial (showing an animal part – heads or hands). In schematic representations, students used more abstract representations like dashes or letters. Some students used drawings as decoration, but reasoned based on schemes so, we considered this as cases of schematic representations. In analysing the participants' discourse by subjects (Bardin, 1977), with a focus on students' reasoning, we came up with two main categories of strategies (working on a random way and on a systematic way) and we further subdivided the second category (working in a systematic way with a single animal first/taking into account the two animals). We come up with these categories working inductively from the data as we explain the ahead in the paper. As in the previous analysis, these categories were built by observing regularities in data. In a second level of analysis, we sought to connect the information provided by both kinds of analyses – regarding representations and reasoning strategies.

STUDENTS' REPRESENTATIONS AND REASONING

Fernanda began by writing on the board the problem for the students to solve it in pairs (one student just worked by herself), providing their response in an A3 paper sheet to be latter collectively discussed by the whole class: "In a farm there are 21 ducks and rabbits. If we count the hands we know that there are 54 in total. How many ducks and how many rabbits are there?" The teacher did a brief explanation about what was intended and the students begun working. When a student pair asked for help, the teacher recalled them the statement of the problem.

Choice of representation

Some students used (informal) pictorial representations but with two levels of detail, representing the whole animal with many features or just representing parts of an animal such as hands or heads. Other students used (pre-formal) schematic representations, either with dashes (to represent hands) or with the first letter of the name of the animal.

Kind of representation	Representations used	Students
Pictorial representation	Detailed drawing of the animals	(1) Maria and Tiago, (2) Vanessa and Eloísa, (3) Joaquim and Francisco
	Simplified drawing of the animals (head or hands)	(1) Renata and Rui, (2) Guida and Júlio, (3) Núria and Daniel
Schematic representation	Scheme with dashes	(1) Dário and Kátia, (2) Ernesto and Patrick, (3) Bruna
	Scheme with letters	(1) Patrício and Sandro

Table 1. Representations used by students in their final work

For example, Maria and Tiago used pictorial representations, drawing successive animals in great detail (figure 1). At some point they begun changing an animal by another. They did not realize that this type of representation does not allow them to get the right answer on time and distract them with too many details, paying attention to irrelevant aspects and not focusing in the important issues at stake.

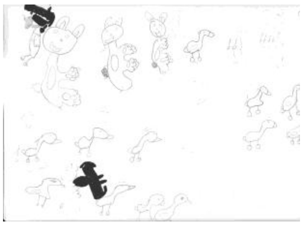

Figure 1. Representation of Maria and Tiago

On the other hand, Guida and Júlio and Daniel and Núria began by making detailed pictorial representations but they concluded that this was not appropriate and made a new schema, just representing animal hands. Daniel and Núria justified their change: "We were already doing drawings, but then [Núria] said: 'Let us erase all because this way we are going to take a long time!' And then we made the hands!". They showed to understand the need to find and adequate representation to solve the task.

Patrício and Sandro made a symbolic representation (vertical computation), that does not represent the givens of the problem (figure 2a). They found a number that added to 21 gives 54, but 54 is the total number of hands, not the number of animals. These two students seem to assume that symbolic representations are the most adequate and allow more efficient reasoning to solve the problem. They use sophisticated representations in an improper way and with which they could not solve the problem.

Figure 2. Representations of Patrício e Sandro: (a) First attempt; (b) Second attempt

Ernesto and Patrick presented a quite elaborated representation, with circled dashes, 2 for ducks (*pato*) and 4 for rabbits (*coelho*) (Figure 3). In this way, as we shall see, they were able to count efficiently the number of hands and the number of animals.

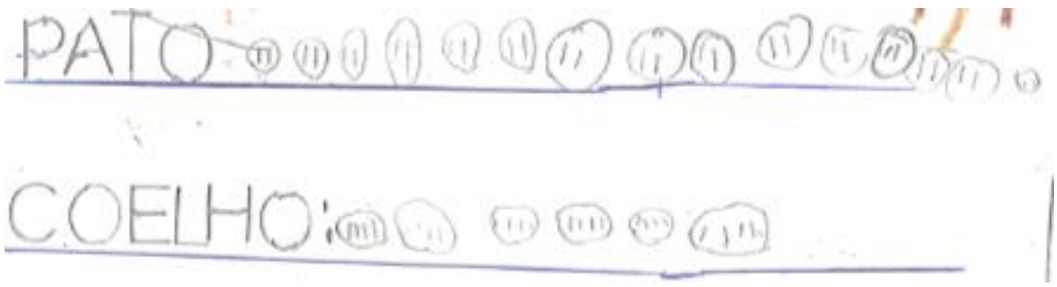

Figure 3. Representation of Ernesto and Patrick

Reasoning strategies and representations

All student pairs solved the problem by trial and error but used two kinds of reasoning strategies: (i) working on a random way (5 pairs); and striving to work on a systematic way, either (ii) taking into account simultaneously the two animals (3 pairs) or (iii) considering a single animal first and then striving to combine the information regarding two animals (2 pairs) (Table 2). We considered that the students used random strategies when they began to solve the problem drawing animals randomly until they meet one of the conditions of the problem. A second group of students strived to work on a more systematic way. They began assuming that they could solve the problem with only one type of animal and drew 21 animals of a particular type. Then, counting the hands, they realized that they had too much (if they drew rabbits) or too less (if they drew ducks) and started to replace some of the animals drawn by the other animal. A third group of students, assumed that the amount of ducks and rabbits was similar and began to draw them in pairs, one of each kind, until they reached the desired number of animals, and, at the same time, they were counting the hands.

Reasoning	Random strategy	Single animal first, then combining two animals	Taking into account simultaneously the two animals
Student pairs	Patrício and Sandro Joaquim and Francisco Maria and Tiago Daniel and Núria Bruna	Guida and Júlio Vanessa and Eloísa	Renata and Rui Ernesto and Patríck Dário and Kátia

Table 2. Reasoning strategies used by students in solving the problem

i) Random strategy. About a half of the student pairs solved the problem in a random way. For example, Núria and Daniel draw ducks and rabbits for a long time. As we mentioned above, at some point they realized that too much detail was not necessary and began to represent the animals in a more simplified way. They searched the solution by trial and error, but in a completely random way. They represented different combinations of animals and counted the number of hands. They were lucky because in one trial, just by random, they found, the solution of the problem. However, the other four student pairs that used a similar random strategy were not as lucky as them and never found the correct solution (Table 2).

ii) Systematic strategy considering two animals. Three student pairs that defined a systematic strategy from the beginning took into consideration the two animals. For example, Dário's last intervention in the following exchange shows in a clear way that his idea is to take into account both animals at the same time:

Dário: (…) How many rabbits there are… And ducks…

Kátia: There are 21…

Dário: 21 what?

Kátia: Ducks?

Dário: No, no, no… There are 21 ducks… And rabbits!! We must know how many ducks there are and rabbits there are, so that we have the twoooo! Get it?

Another student pair, Renata and Rui, explain to the teacher how they considered the two animals and, at the same time, counted the number of hands (Figure 4):

Renata: We drawn a duck and a rabbit and drawn another duck and another rabbit… We went doing this way…

Teacher: You draw a duck, a rabbit, a duck, a rabbit, a duck, a rabbit… Always…

Renata: Yes… And then… We stopped a little, and we counted… Counted hands and how many animals we had to do and… Counted hands to see if we had to erase or add more, or erase and let it be like this.

Figure 4. Representation used by Renata and Rui

Renata and Rui started working in an inductive way, trying out specific cases. They began by assuming that the number of ducks and rabbits should be similar, so they draw one after the other. At some point, they evaluated their work and realized that although the number of hands was high (already 50), the number of animals was much below of what was needed. So they had to make some changes and erased rabbits putting ducks on their place, so that they got 21 animals. Doing this, they ended up with the number of animals required, but they were surprised to verify that the number of hands was not enough. They were puzzled, but the teacher helped them to understand that they needed to replace back some ducks by rabbits.

Teacher: You have 21 animals, but you are short of hands… And now?

Renata: I have to put some rabbits and… (…)

Teacher: You put more rabbits? But if you draw more rabbits, there will be more than 21 animals, or not? How shall we do that? (…)

Renata: I will erase two ducks and draw one rabbit!

The statement that two ducks are equivalent (in the number of hands) to one rabbit is an important generalization. The idea that this inference (drawn from the givens of the problem in a deductive way) may be used to solve this problem is an important insight. Having done that, the students further conjecture that they would not need to do great changes and they would just need to replace 4 ducks by 2 rabbits. That was close to what was required, but still not the solution of the problem. Supported by the questions of the teacher, the students finally concluded what animals they still needed to draw to solve the problem:

Teacher: So?

Renata: We erased 4 ducks and made 2 rabbits!

Teacher: Yes… And now? How many hands we have and how many animals we have? Rui?

Rui: Now we have… 1, 2, 6 (counts hands one animal by one) 50!

Teacher: 50 hands… I want 54! We are almost there! How many animals we have?

Students (Count one by one and answer): 19.

Teacher: And how many animals we must have?

Renata: 21…

Teacher: You have 19… (…) How many animals are missing?

Renata: We need 2 animals more!

Teacher: And how many hands?

Renata: 50… 54, so we need to draw 2 ducks!!

Teacher: Why?

Renata: Because… 2 more 2 is 4 and we have 50 [hands] and we need 2 more animals and the ducks have 2 hands… I think it will work!

The drawing with the 19 animals that allows for the counting of the 54 hands provides the justification that this is a solution to the problem. It is a solution by exhibition, since the students constructed an object that satisfies the conditions of the problem. Later on, Renata provides a very explicit explanation of the relationship between the hands of ducks and rabbits, which may be regarded as the key generalization in this task:

Renata: If two ducks have four legs and one rabbit has four legs… And we had too much animals… We erased two rabbits… Oh! No! [We erased] two ducks and we draw one rabbit!

Ernesto and Patrick had an elaborated representation with circles and dashes (figure 3) and explained to the teacher their strategy to solve the problem: "We have made schemes by a trial strategy (…) We though 10 ducks and 11 rabbits. But then we saw that does not give us 54 hands and we did it again" (Patrick). They also used an inductive approach, trying out cases, and figured out the equivalence between two ducks and one rabbit.

iii) Systematic strategy beginning with one animal. Other students followed a different strategy, seeking first to fulfill one of the conditions of the problem and then doing attempts to replace some animals by the other. For example, Guida and Júlio began by drawing just ducks:

Guida: We were doing… 2 [Two hands – a duck] up to 19 [animals], but then it doesn't give us 54 hands!

They concluded that if they kept drawing only ducks they would have more animals than required. By realizing the key generalization that one rabbit is equivalent to two ducks they quickly figured out how to solve the problem replacing two ducks by one rabbit or vice-versa until they had the right number of animals and hands. Therefore, they began by representing only one kind of animal, but at some point, they evaluated the situation and they understood that they needed to combine the two kinds of animals and to use a systematic strategy of replacing an animal by another until they got the correct solution.

Vanessa and Eloísa began by drawing just rabbits, until they got 21 animals but they realized that they had much more hands than required. They randomly erased several rabbits and started drawing ducks (but not replacing one by one). They did this several times, until they got 54 hands. At this moment they realized that they had only 19 animals. With the help of the teacher, they finally found the solution:

Teacher: (counts animals with fingers one by one) 19! (…) But you need 21! And the 54 hands? (the students shake their heads meaning yes)

Teacher: What are you going to do?

Vanessa: We need to replace…

Teacher: To replace why? And by what?

Eloísa: Rabbits by ducks…

Teacher: Huummm… How many rabbits by how many ducks?

Vanessa: I am going to take… I am going to take a rabbit and then I will put… (looks at the duck) I am going to put two, because they have two hands!

After working at random for some time, these students finally noticed that a rabbit and two ducks have the same number of hands (the key generalization) and that they may change two ducks for one rabbit, keeping constant the number of hands.

In this way, the student pairs that define a systematic working strategy, either considering one or two animals, and that at some point evaluate their work and discover relationships among different elements (especially the generalization that two ducks have as many hands as one rabbit) are those who more quickly solve the task.

CONCLUSION

Solving this problem involves three crucial moments: (i) understanding and representing the conditions, (ii) defining a reasoning strategy, and (iii) applying and monitoring that strategy. First, the students have to understand the problem conditions and make a suitable representation. To understand the statement of the problem it is not easy for some of the students at this level. Some use pictorial (informal) representations, drawing animals in detail, a very slow, distractive and inefficient representation process. Other students use schematic representations such as dashes and hands (pre-formal), without spending time on details, what makes the solution quicker and allows them to focus their attention in important elements of the problem. Some represent ducks and rabbits in different lines or blocks, or sequences of duck-rabbit pairs, for easier counting. Others, to make sure that the number of animals is equal or easily comparable, draw them alternatively. Still some students use symbolic (formal) representations that do not represent well the problem, naturally with no success.

In a second moment, based on their representation, the students define a solving strategy that in this case is always by trial and error. We must note that the problem may be solved algebraically with a system of four equations and four unknowns ($4h=R$; $2k=D$; $h+k=54$; $R+D= 21$, with h and k being the number of hands of rabbits and ducks and R and D the number of rabbits and ducks) but grade 3 students cannot do it this way. They have to follow an inductive approach, trying out different combinations of animals. The key to reach to a solution is to design a strategy of working in a systematic way, either taking into account simultaneously the two animals, or considering a single animal first and then combining the information regarding the two animals. However, to solve the problem the students need to recognize the four relations above. Starting with both animals means to take into account the equation $R+D=21$, then changing the value of one variable (the other just follows from that) and seeing what happens to the sum $h+k$. Starting just with one animal is similar, except that we begin with $R=0$ or $D=0$. In both cases, it is critical to notice that in replacing one rabbit by two ducks $h+k$ remains constant (which is the key generalization in the problem) whereas $R+D$ increases 1 unit. This may lead to formulate an hypothesis that replacing one rabbit by two ducks or vice-versa will solve the problem. All steps done by students may be justified by the givens of the problem and the allowed operations. The justification why 6 rabbits and 15 ducks is the solution is done by exhibition, counting the corresponding number of hands.

The students that use pictorial representations tend to follow a random strategy. The students that use symbolic representations also have trouble because the representations that they choose are not adequate to consider the problem conditions. The students that use schematic pre-formal representations (Webb et al., 2008) that salient important aspects of the problem (number of hands of each animal) and omit irrelevant features (such as the physical aspect of the animals), are those who formulate the most efficient solution strategies. The fact that students used a wide variety of representations with distinct potential for solving the problem, shows that the teacher may assume an important role, helping the students to use increasingly sophisticated representations with understanding. The difficulty of the students in keeping track of the conditions of the problem and in reasoning in a systematic way also suggests the critical role of the teacher in dialoging with students. This underlines the need to further study how teachers may support their students in learning to use mathematical representations and to develop their reasoning processes.

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