THE (RELATIVITY OF THE) WHOLE AS A FUNDAMENTAL DIMENSION IN THE CONCEPTUALIZATION OF FRACTIONS

Nadine Bednarz & Jérôme Proulx

Université du Québec à Montréal, Canada

This paper is focused on a fundamental element in the conceptual development of fraction, the part-whole sub-construct. This focus is grounded in a research project with teachers in which 3 days were spent on fractions and where issues related to the (relativity of the) whole*, or what was called "the referent," occupied a significant space of learning. Through outlining how teachers' understandings evolved in the sessions, we illustrate the fundamental role of* the (relativity of the) whole *in learning about fractions, and highlight the complex ramifications that underpin its learning.*

INTRODUCTION

In the teaching and learning of mathematics, fractions have long been seen as one of the most difficult concepts to understand (see e.g. Ball, 1990; Boulet, 1998; Charalambos & Pitta-Pantazi, 2007; Tobias, 2012). Following Kieren (1976) and Behr et al. (1983), these studies confirm the complexity of this concept, which is not a single but mainly a multifaceted one. This complexity of fractions as an amalgam of interrelated sub-constructs has been first developed by Kieren (1976) and extended by Behr et al. (1983), creating a model of different interpretations of fractions, mainly the part-whole, ratio, operator, quotient, and measure, integrated and linked to operations, fraction equivalence, and problem solving (cf. figure 1).

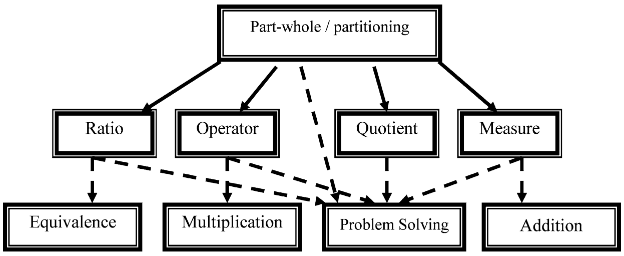


Figure 1. Behr et al. (1983) model (taken from Charalambos & Pitta-Pantazi, 2007)

While focusing on the importance of developing flexibility to handle these different interpretations, to move from one to the other in operations and problem solving (see e.g. Charalambos & Pitta-Pantazi, 2007), this model outlines that the part-whole sub-construct is of fundamental importance in the process of fraction understanding.

One aspect we focus on in this paper, and that we argue is of paramount importance in that part-whole sub-construct, even vital to its understanding, is *the (relativity of the) whole*; what came to be called the "referent" by the teachers engaged in the project. Below, we show how this *(relativity of the)* whole is of particular importance and explain what we mean by it. Then, after having explained the study's objectives and methodological considerations, we detail its intricacies through analyzing its evolving meaning within a group of in-service elementary teachers.

PART-WHOLE SUB-CONSTRUCT AND (RELATIVITY OF THE) WHOLE

The (relativity of the) whole, albeit not explicitly outlined in Behr et al. model, has been alluded to by a number of researchers, particularly in studies involving teachers (e.g. Ball, 1990; Simon, 1993; Schifter, 1998). For example, in Simon's study, prospective teachers were presented with a division of fraction problem for finding the number of cookies 35 cups of flour make if one cookie needs 3/8 of a cup. In their answers, numerous teachers who arrived at a remainder of 1/3 defined it as the remaining flour instead of seeing it as 1/3 of what it takes to make a cookie. In similar ways, Tobias (2009, from Tobias, 2012) explains that when her prospective teachers were asked to find "how many 1/4 pound servings of dough can be made from 1 7/8 pounds of dough" they answered 7 1/8, saying that 1/8 referred to the whole pound instead of a serving of dough. This raises the issue of referring to the *proper whole* when discussing fractions in the part-whole sub-construct.

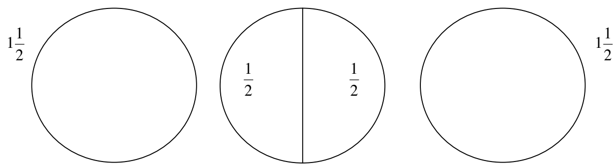
As well, more recently, Tobias (2012) developed an entire paper on the notion of the whole of a fraction, addressing it through issues of language use. Through highlighting various difficulties elementary teachers experience with fractions (e.g. Ball, 1990; Simon, 1993), she focuses on the importance, in fractions teaching and learning, to well conceptualize the whole of the fraction in order to contextualize situations, to understand the procedures to use and to interpret various solutions. She argues that a number of difficulties lived by learners with the notion of whole comes from language difficulties or from an inappropriate use of language in defining wholes. She gives the following example (p. 2):

> In the context of subtraction, problems such as 3–2 can be stated as starting with three objects and taking away two of them. When the situation involves fractions, such as 3–½, it is incorrect to interpret this as starting with three objects and taking away half of them.

Even if we agree with Tobias' argument that language issues are of importance, we believe that there are deeper ramifications that are at play in understanding the whole of a fraction. One of these deep ramifications that we outline in this paper is related to the whole itself. We address it as *the (relativity of the) whole*.

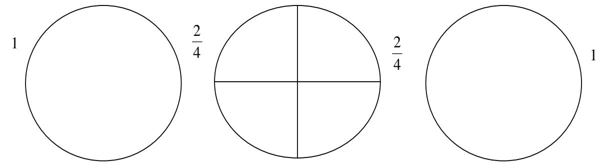
In order to clarify what we mean by *the (relativity of the) whole* and to see its importance, a Grade-4 classroom vignette is used, taken from a previous study on fractions. In this vignette, students had to solve the following problem: "Share three pizzas equally between two children". They had to find as many ways as possible to solve it (with three circled pizzas was drawn on the board for visual support). Here is an extract of the classroom discussion related to its solving.

> *Marlene*: One child has a pizza, the other one also has an entire pizza, then one child has half, and another half. A pizza and a half each (she draws it on the board).



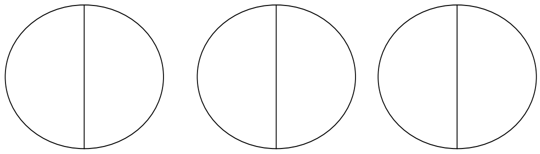
Teacher: Ok, does it work? [pupils: yes!] Does anyone else have a solution?

Manon: There's one child that has one pizza, the other one has another pizza. I split the remaining one in four, one child gets two fourths and the other one two fourths. One child has a pizza and two fourths and the other one a pizza and two fourths.



Teacher: Ok, another solution?

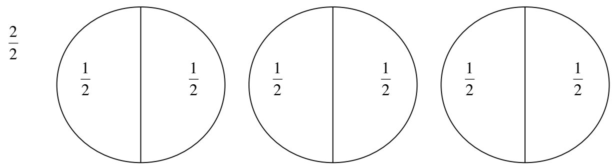
Veronique: I separated my pizzas like this: half of a pizza for a child, the other half for another child, another half for the first child, another half for the second, another half for the first, another half for the second; which means each child has three sixths.



Teacher: What do you think? Is this right?

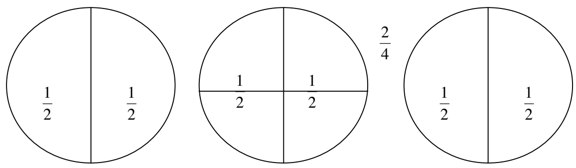
A student: No! Half plus half, that's two halves.

Teacher: Two halves (writing it on the board). Does this mean two over two?



Martin: No, one over two and one over two, that's two over four.

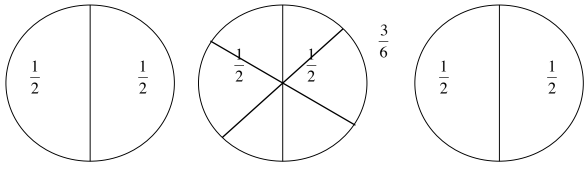
Teacher: You mean I divided a pizza in four, and I took two pieces.



Michel: No, it's not two over four [referring to Martin solution] it's one pizza and a half, because it's two halves together, one pizza plus one half, that's one pizza and a half.

Veronique: Yes, but three sixths [returning to what she said previously], that works…

Teacher: Does this mean we divided a pizza in six pieces, to find something like that?



Veronique: No, in the end it's all the same.

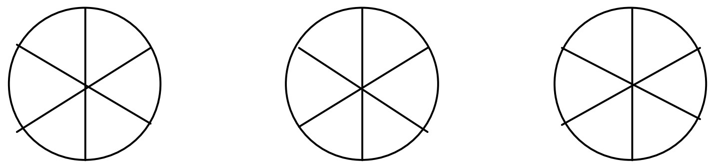
Teacher: You mean we divided by the same thing, and have the same parts of the pizza?

Veronique: We have the same result.

Teacher: Ok! Let's go back to 3/6, how do I get this? What bothers me is the three sixths.

Gabrielle: Oh, I know! There's three because there's six in total, six pieces…

Teacher: Oh, as if one pizza was divided in six pieces. Alright, it's like the pizzas were divided in six pieces, right?



Gabrielle: No, we need to divide it in two, it makes three pieces each… All the pizzas, all of them together are divided in six parts.

Another student: Does this happen by chance, because you know, there is one out of two, you take one pizza and each pizza you take one out of two, and one out of two, and one out of two, it makes three sixth.

Of interest in this teaching vignette is not if the teacher reacted well to students' answers and questions or if students themselves understood well the concept of fraction. Of significance, within the myriad of answers given, is the possibility of the validity of these answers and on what grounds. When Manon says 1 and 2/4, the whole she refers to is one of the pizzas, her answer being 1 pizza and 2/4 of one pizza. When Veronique says 3/6, the whole she refers to is the entire three pizzas, making her answer 3/6 of the pizzas. When Martin adds ½ and ½, that gives him not 2 over 2 but 2 over 4, is he wrong or could he be adding ½ of one pizza with ½ of one pizza which gives him 2 of those halves out of 4 halves? And, again, could the student question about the coincidence of ½+½+½ giving 3/6 be along a change in the whole to represent the total? Those for us are important questions and are not answerable by a right/wrong dichotomy. This view of *the (relativity of the) whole* is closely aligned with Schifter's (1998) view, who for example queries "How can that piece of cake be 1/2 and 1/4 at the same time?", a kind of question frequently heard. In the part-whole sub-construct, fractions always need to be related to a whole. And, that whole can change, leading to the mathematical validity of what is advanced. For example, the fraction ¼ alone cannot "be", as it needs to be ¼ of something. And, it is in relation to these "somethings" that comparisons, e.g., can proceed. If one asks "which is the biggest ½ or ¼?" there can be a variety of answers: if the same whole is considered, ½ is bigger; if different wholes are considered for each fraction, then ½ can be bigger, smaller or equivalent to ¼. This *relativity of the whole*, mainly through relating the fraction to different possible wholes, is of fundamental importance in the mathematical reasonings about fractions. We document in this paper how this issue of *the (relativity of the) whole* took form, through analyzing the evolving interpretations of a group of elementary teachers in different tasks.

RESEARCH OBJECTIVES

Though a number of studies (e.g. the ones mentioned above) have reported on issues of *whole* in teachers or students' understanding of fractions, few have detailed the ways in which this notion develops in learners/teachers. In this paper, we aim at extending this literature by documenting in detail the evolving interpretation of this notion in learners, here being practicing elementary teachers. In examining the construction of *the (relativity of the) whole*, as one central aspect of the part-whole sub-constructs, our report offers an analysis of different meanings of this fundamental aspect related to fractions (the whole) as well as of the intertwined meanings of the fraction concept itself; aspects of central importance for better understanding the phenomenon of teaching and learning fractions. In that sense, the paper is oriented by the following questions: (1) in what ways do learners, here practicing elementary teachers, understand the notion of whole? (2) In what ways does this notion evolve, develop? (3) In what ways does the notion of (relativity of the) whole relate to other sub-construct of fractions?

METHODOLOGICAL CONSIDERATIONS

A group of 10 elementary teachers (Grade-4 to 6) participated in a 2-year professional development (PD) research project. This PD-research project intertwined professional development and research concerns, and can be related to a teaching experiment (Steffe, 1983) through its preoccupation for documenting students' conceptual development overtime. In the teaching experiment methodology, the researcher – who is also the teacher – builds models of meanings developed-in-action by students, and confronts, during the teaching episodes, these models to the reasoning's and actions mobilized by students in new situations (leading to a continual restructuration of the models built). This methodology helps to understand a conceptual development in all its complexity and over time. In this research project, our preoccupation is similar. The intention is not to develop an in-service project for showing its potential in terms of learning outcomes. This PD-research project is an occasion to document the conceptual development (in this case of elementary teachers) about a specific arithmetic content. As teacher educators-researchers, we designed and conducted sessions, participating in the development of the mathematical understandings occurring in them, pushing their elicitation and modeling the conceptual development (as Steffe did).

The initiative is structured around day-long monthly sessions (15) during 1 ½ school year, all of them being videotaped, to keep a record of the sessions' unfolding, and a researcher journal was kept about salient events and reflections these provoke. The sessions activities revolved around "mathematics" tasks for teachers to engage with, articulated on their practice, about different mathematical topics [fractions, division, measurement (perimeter, area, volume), decimals numbers]. Teachers were invited (in small groups followed by plenary discussions) to engage with those mathematical events and to explore/discuss/make emerge the mathematical ideas inherent in these tasks. The focus here is on the first block of 3 sessions on fractions, where tasks were about the part-whole understanding of fractions through partitioning contexts.

The data analysis procedures adapted the approach proposed by Powell, Francisco, & Maher (2003), and focused on an *emergent* coding of data. Oriented by the notes from the research journal to pay attention to specific events of significance that happened in the sessions, the first stage of analysis involved (re-)becoming familiar with the sessions in full, viewing the tapes in their entirety to get a sense of their content. Specific events of importance were pointed to for orienting the continuity of the data analysis. In the second stage, the video data were described through writing brief, time-coded descriptions of each video's content and grouping it in "events". In stage three, data were reviewed anew to explore more precisely possible "significant events", which led, in stage four, to precise data transcriptions (verbal, gestures, drawings on the board, etc.) regarding these events. Each event was then analyzed in details and related to the previous and following ones to develop an evolving pattern concerning the (relativity of the) whole understandings. We outline these below.

ANALYSIS OF EVOLVING MEANINGS OF THE WHOLE OF FRACTION

The relevance of the whole, that came to be called the "referent" during the sessions with teachers, emerged progressively as a significant issue to consider in fraction teaching and learning. We present some of the tasks worked on below, and the different meanings given to the "referent" through these tasks as they emanate from the analysis. Because of space constraints, only a synthesis of the different conceptualizations, and their evolving meanings, are outlined.

An evolution of the referent through various problems

As teachers explored this first problem about "John and Mary" (with sample solutions), a number of meanings about the referent were engaged with.

John and Mary problem. John and Mary each have pocket money. Mary has spent ¼ of her amount and John ½ of his. Who spent more money, John or Mary?

Student solution 1:	*Student solution 2*:
John because ½ is more than ¼. Ex.: ½ of 16 = 8 but ¼ of 16 = 4	Mary ¼ John ½ = 2/4 John spent more and Mary less

(1) **Referent absent from the solving process**. In some explanations of teachers (e.g. below), the referent is *not* present in the way they make sense of the solutions, highlighting at the same time that these fractions are considered as "absolutes".

M.: We don't know how much money they [John and Mary] have, but it is not important. This child understands well the meaning of fractions, that ½ is more than ¼. And, he offers an example to confirm it.

One could be tempted to say that fractions are here treated as numbers, an important meaning in the learning process on fractions (i.e. from elementary to secondary school, from natural to rational numbers). However, the analysis of the transcripts shows that fractions were not explained as numbers but mainly taken for granted as a thing in itself, an evidence (½ is simply a bigger fraction than ¼).

(2) **Contextual referent**. Another meaning emerged, in interaction with the previous one and influencing it, in which the referent is considered. Even if in this case the referent emerges as important, this new meaning does not really affect the meaning of the concept of fraction itself, but is mainly related to the context of the problem.

> *Interaction1*.
> G.: My students, on the contrary, could have said that we can't answer this question because we don't know how much money they have at the beginning. Then if one has several millions dollars and the other has little money, the ¼ could be more.
>
> M.: Right. If I have 100$ and you have 10$, even if you spent half I stay richer. Our group did not see it like that.
>
> *Interaction 2*.
> M.: For the meaning of the fraction, this student would get a good mark. For me, he understands the meaning of a fraction, that ½ is more than ¼.
>
> J.: But, if he does not mention that there is no referent, no amount of dollars, then we can't say that his solution is good. […] If the child does not say that it depends of the beginning amount, I can't give him all the marks, since it could be different amounts.

In this case the context of the problem creates the necessity of considering *the referent* (the amount at the beginning) but this contextual referent does not affect the meaning of the fraction (½ remains bigger than ¼). This referent stays independent of the fraction, only impacting the resulting amount where the fraction continues to be treated as an absolute.

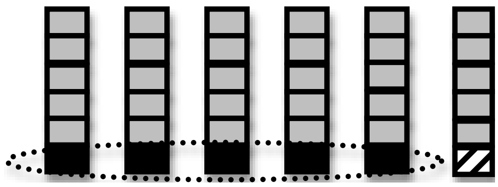
(3) **Fixed referent**. Later on, the consideration of the referent influenced and evolved toward the meaning of fraction itself, for which not only the context provoked its consideration. This referent is, at this moment, conceived as fixed, determined in advance, so that it is possible to operate on fractions and conclude when its value is known *a priori*.

> M.: If we want to practice the meaning of fraction, then give children a referent, and stop playing with it. Hence, it would be necessary in the problem to say that each person has the same amount. Our group started with the fact that each had the same amount.
>
> Mi.: We could also, when there is missing data, work with situations where there is the same amount and other situations where there are different amounts.

Even if there is a relation linking the fraction to a whole, there is a need to make this referent fixed, determined, so that one does not have to consider it afterwards when operating on fractions. This referent can vary from one situation to the next, but it needs to be fixed for each of those situations (like one fixes the parameters of a function when treating a specific case).

(4) **Relative referent**: This previous meaning (fixed referent) contrasts with another one (relative referent), that appeared much later in the sessions. We present here some interactions that occurred regarding the "ribbon problem" (Schifter, 1998)

Mali has 6 meters of material. She wants to make ribbons of 5/6 meters for a birthday at school. How many ribbons can she make and how much material will she be left with?



Student solution 1: 7 and 1/5

Student solution 2: We can make 7 ribbons

Student solution 3: 7 and 1/6 of a ribbon

Student solution 4: 7 and 1/7

A.: [discussing the solution 1] I mean the ribbon is finally 5/5.

Mi.: No! 5/6.

A.: No, no, no. In the end, it becomes 5/5 [M.: Oh my god!]

A.: And then the last piece remains a part of my whole, it remains 1/5.

Mi.: Yes, I understand what she means. We change the whole!

The fraction is here clearly related to a whole, and relative to this whole : the same piece can take several values (e.g. 1/5, 1/6, 1/36) depending of the whole considered (it was not the case when the referent needed to be fixed, determined). In this case, somehow the referent is seen as variable, whereas in the previous case it was conceived as parametrical (that is, changing but to be fixed for a specific problem).

This sophisticated meaning about the referent was developed, explained and refined on a long-term basis, through various tasks and interactions between participants. However, along this process, other interrelated meanings emerged. We present below, through short excerpts, three other meanings given to the referent.

(5) **Referent as pictorial representation**. For a number of tasks where students used pictures in their solutions (e.g. the ribbons problem), the referent became linked to a pictorial representation, seen as the whole. In this case of partitioning, the part needed to be completely included in the whole (in the picture of it). We found here an idea of inclusion according to which the parts are considered as elements of the whole. This is linked to a conception of fraction as a comparison between the part (a number of pieces) to the whole (the total number of pieces in which the picture is partitioned).

Interaction 1: on a problem involving a comparison of 5/6 and ¾

S.: When we have to compare fractions, we know what is the partitioning. When you understand the partitioning of the fraction, you make a drawing, dividing it e.g in four and you take 3 of them.

Interaction 2: on the "ribbons problem", concerning the 7 and 1/5 solution

S.: It can't be 1/5 of ribbon. A ribbon is 5/6, so it is 1/6 of ribbon, you can't say 1/5. The remaining piece is not inside the ribbon.

It is the picture, acting as the whole, that impregnates the meanings of the referent and the fraction itself. The pieces considered must be "inside" this whole, being part of it. This referent is dependant on the drawing done, based on a view of fraction as a comparison between a part and the whole in which it is included.

(6) **Referent as common denominator**. Sometimes the referent becomes the common denominator, and is seen as the basis on which to compare fractions (as we see below in a problem involving a comparison of 5/6 and ¾).

S.: We need to compare with the same whole, oranges with oranges, apples with apples.

M.: There are quarters and sixths, so it is apples and oranges.

A.: We need to make it on twelve.

M.: You have to put them on the same denominator to compare, so it becomes our referent. We make it in twelfths and this is the referent.

Of interest here is that through making the common denominator the referent, implicitly teachers are transforming the comparison of fractions to a comparison of natural numbers [(5/6=10/12 and ¾=9/12, compare 10 (twelfths) and 9 (twelfths)].

(7) **Referent as a number linked to a fraction operator**. In the "ribbons problem", the following discussion happened concerning the remaining piece being 1/6.

N.: It means that it is 1/36 of 6 meters of material. But it can be also 1/6 of one meter.

S.: Because we can simplify the fraction, 1/6 x 36.

Teachers here use an operator to arrive at 1/6 of a meter (1/36 of 6m = 1/36 x 6 = 6/36 = 1/6). They thus agree that the same piece of material can have different values (1/36 of 6m, 1/6 of 1m), *through the operation*. Hence, it is not the consideration, for the same piece, of different wholes that leads to this acceptation, but a multiplication and simplification on numbers (fractions are here considered as operators).

DISCUSSION AND CONCLUDING REMARKS

The varied meanings given to the referent in the reconstruction of the conceptual development over the sessions show a complex picture of its evolution (even if more excerpts would be needed, but space constraints forbids). Different meanings emerge, interact and influence each other, taking form, being explained and refined, etc., all along the process of exploring and discussing the various tasks. These meanings, for the same or different persons, oscillate between one another; showing that these do not follow a linear path. These interacting aspects illustrate well the richness of this development, showing the complexity of the learning of fraction in partitioning contexts. It shows that the part-whole sub-construct is a complex one, involving not only an understanding of what many researchers refer to the partitioning scheme with its different components (Charalambos & Pitta-Pantazi, 2007, Boulet, 1998), but also a rich conceptualization of the whole (reasoning on the fraction in relation to the whole, seeing the relative value of the part, re-organizing the whole, etc.).

Even if many researchers point to the fundamental dimension of the sub-construct part-whole in the process of understanding fractions, the place occupied by a reflection on the meaning of the whole in this conceptual development has not been amply considered. The understanding of the "the whole" acquired a broader significance through this research project. This question of referent is more than a question of language as Tobias (2012) e.g. would have it. The notion of defining the whole is related, as we have seen, to the underlying meaning of the concept of fraction itself in a part-whole context. This "change" or "relativity" of the chosen whole is central in relation to comparison, operations (e.g. ¾ x ½, where ¾ is seen as having ½ as its whole, something often seen as "taking a part of a part of the whole"), equivalence and problem solving. In addition, even if we haven't highlighted it here, the various meanings developed about the referent are often nested with other sub-constructs. For example, issues of "referent as number on which to operate" is closely linked with the sub-construct "operator" of the fraction, as well, important issues concerning ratios were dealt with through the sessions. Those issues are to be developed in another paper.

REFERENCES

Ball, D.L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, *21*(2), 132-144.

Behr, M.J. et al. (1983). Rational number concepts. In R.A. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-126). Academic Press: New York.

Boulet, G. (1998). Didactical implications of children's difficulties in learning the fraction concept. *Focus on Learning Problems in Mathemaitcs*, *20*(4), 19-34.

Charalambos, Y.C., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics*, *64*, 293-316.

Kieren, T. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R.A. Lesh (Ed.), *Number and measurement* (pp. 101-144). ERIC: Columbus.

Schifter, D. (1998). Learning mathematics for teaching: From a teacher' seminar to the classroom. *Journal of Mathematics Teacher Education*, *1*(1), 55-87.

Simon, M.A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, *24*(3), 233-254.

Steffe, L.P. (1983). The teaching experiment methodology in a constructivist research program. *Proceedings of ICME-4* (pp. 469-471). Boston: Birkhäuser.

Tobias, J.M. (2012). Prospective elementary teachers' development of fraction language for defining the whole. *Journal of Mathematics Teacher Education*.