MATHEMATICS TEACHER'S SPECIALIZED KNOWLEDGE. REFLECTIONS BASED ON SPECIFIC DESCRIPTORS OF KNOWLEDGE

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We present some reflections on mathematical knowledge for teaching arising from the descriptors proposed by Sosa (2011) and linked to the model of mathematics teachers' knowledge constructed by Ball, Thames and *Phelps (2008). The former leads us to reflect on the model and to adopt the proposal of Mathematics Teacher's Specialized Knowledge so as to make progress in describing the mathematical knowledge brought into play when teaching. This suggests the need to scrutinise the model carefully and to refine its characterisation. To do so, we embark on a search for evidence which allows the incorporation, integration and interconnection of aspects of knowledge apparently unrelated in the model of Mathematical Knowledge for Teaching (MKT).*

Key words: Mathematical knowledge for teaching, mathematics teachers' specialised knowledge.

INTRODUCTION

In this paper we discuss the theory underlying the different domains which comprise the construct of mathematical knowledge for teaching developed by Ball, Hill, and Bass (2005) and Ball, Thames, and Phelps (2008), a model widely used in Mathematics Teaching research for the purposes of teacher training. The model devised by these authors allows teachers' knowledge to be studied from observations of classroom practice in primary education. Building on the work of Shulman (1986), it focuses on mathematical content through the categories of content knowledge and pedagogical content knowledge.

[1] This article represents part of the studies into teachers' knowledge by the SIDM group (from the Spanish 'Research Seminar into Mathematics Education') based at the University of Huelva, Spain. It comprises the following researchers: José Carrillo (coordinator), Nuria Climent, M. Cinta Muñoz-Catalán, Luis C. Contreras, Miguel A. Montes, Álvaro Aguilar, Dinazar I. Escudero, Eric Flores and Enrique Carmona (University of Huelva), Pablo Flores, Nielka Rojas and Elisabeth Ramos (University of Granada, Spain), C. Miguel Ribeiro, Rute Monteiro and C. Susana dos Santos (University of the Algarve, Portugal), Leticia Sosa and José L. Huitrado (University of Zacatecas, Mexico), and Emma Carreño (University of Piura, Peru)..

We discuss the model of MKT in terms of the general descriptors of mathematical knowledge defined by the research group headed by José Carrillo at the University of Huelva, Spain. Amongst the results of this line of research are the studies by Sosa and Carrillo (2010) and Sosa (2011), which report on an interpretative study into the MKT displayed by two teachers working on the topic of matrices at Spanish baccalaureate level. For more specificity of mathematical knowledge for teaching, Sosa identifies and builds indicators or descriptors of knowledge or specific skills of the teacher while teaching (p.54). In order to specify more precisely mathematical knowledge for teaching, Sosa identifies and draws up a set of descriptors and indicators of the specific knowledge and skills deployed by teachers in their work (p.54). These studies suggest that further work on the features demarking the subdomains of mathematical knowledge would be beneficial. Sosa's study, dealing with general aspects of mathematical knowledge, is a good starting point.

This paper aims to discuss the knowledge domains described by Sosa (2011) and to scrutinise the model in order to develop its characterisation. The need to illustrate a new system of organising teachers' mathematical knowledge, drawing on MKT and being made operative by means of practice-based descriptors, leads to a new organisation which lays emphasis on the specific features of this knowledge in relation to teaching.

Below we discuss aspects of MKT in terms of Sosa's indicators, and consider how the new perspective might deal with certain problems arising, with the aim of refining the model so as to better understand the knowledge teachers display as they go about the work of teaching.

THE RELATION BETWEEN MKT AND KNOWLEDGE DESCRIPTORS

Mathematical knowledge for teaching is understood by Ball and her collaborators as the specific knowledge required for teaching mathematics. This model builds on the work of Shulman and collaborators several decades previously. Furthermore, as an emergent contribution of the qualitative analysis of the classroom practice of various practising and trainee teachers in the United States, it confirms the specialisation of the knowledge required for teaching mathematics, distinguishes components of this knowledge (in terms of domains, subdomains and descriptors), and includes the subdomain of specialised content knowledge, which is fundamental to the work of teaching.

With this in mind, the group based at the University of Huelva aims to explore the MKT model so as to better understand and identify its components. Hence, in 2001, the group compiled a list of 100 general descriptors for mathematical knowledge which enable elements to be differentiated and features of the observed knowledge to be established in the practice of teaching. These descriptors correlate with the components defined by Ball et al. (2008), and are carefully phrased to capture each particular characteristic pertaining to the various knowledge domains, such that items representing evidence of aspects of knowledge in the MKT model can be built up, integrated and inter-related. Below, we describe how the MKT knowledge subdomains relate to some of the descriptors proposed by Sosa (2011) and discuss several issues arising.

The subdomain *Common Content Knowledge* (CCK) is based on the notion of encapsulating the mathematics that anybody making use of the subject might know, such as might be the case in using definitions, rules, properties and theorems associated with a specific topic (*CCK1*), using mathematical notation, and understanding the importance of an item (*CCK2* and *CCK3*), and knowing how to apply mathematics and do demonstrations (*CCK4* and *CCK5*). Complementing the foregoing, we would add that is a kind of knowledge that teachers need, and although other professions might draw on it, too, it forms an integral part of what makes mathematics teachers specialists.

The subdomain *Specialised Content Knowledge* (SCK) refers to a deeper and more thoroughgoing knowledge of mathematical content, and includes understanding the significance of concepts (*SCK1*), knowing the unseen steps behind procedures (*SCK2*), intuiting the root of pupils' mathematical errors (*SCK4*).

The descriptors corresponding to *Horizon Content Knowledge* (HCK) refer to associations between concepts, relationships between general and specific content (*HM1* and *HM2*), and awareness of interdisciplinary applications (*HM3*). In the case of descriptors referring to understanding how one item relates to another that comes before or after it in the curriculum, the knowledge that is brought into play concerns curricular issues and the sequencing of the subject, as the teacher has to know the contents both previous and subsequent to any particular item being taught.

Pedagogical Content Knowledge (PCK) combines knowledge of teaching with knowledge of mathematics (Ball et al., 2008, p. 401). This subdomain concerns aspects such as those embodied in descriptors *PCK30*

and *PCK31*, respectively: "*Knowing how to introduce a new concept by relating it to concepts studied previously*," and "*Knowing different ways of introducing a mathematical topic through some information or brief historical background about it; or knowing how to contextualise a topic through a brief anecdote or historical background*." These descriptors illustrate that PCK is a subdomain which implies learning mathematics with meaning. Indeed, sometimes descriptors tend to be associated with features of learning mathematics such as "Knowing which exercises to leave the pupils for homework."

With respect to *Knowledge of Content and Students* (KCS), there are two sets of descriptors, those referring to general pedagogical knowledge, and those referring to knowledge about the students' interaction with mathematics. The first of these sets represents knowledge that mathematics teachers probably need, and hence the descriptor (*KCS2*), "*Understanding the needs and difficulties of students with mathematics*." The second group is based on understanding how pupils assimilate and apply material, as can be seen in the descriptor (*KCS3*), "*Anticipate the misunderstandings that might arise with specific items being studied in class*." The descriptors in this subdomain enable us to distinguish when knowledge involving aspects of learning mathematics is being deployed, which is an appropriate starting point for describing this subdomain.

Finally, regarding *Knowledge of the Curriculum* (KC), Sosa (2011) sets out three descriptors describing the organisation of content in textbooks (*KC1*), the prior and subsequent treatment of an item (*KC2*), and content deriving from teachers' institutional environment (*KC1)*. These descriptors also teachers' critical responses to the established objectives and standards to be noted.

TOWARDS THE SPECIFICATION OF MATHEMATICS TEACHERS' KNOWLEDGE

Taking into consideration the conditions for distinguishing the subdomains of MKT and attempting to find those defining features which specify mathematics teachers' mathematical knowledge, we find it necessary to place mathematics at the hub and focus attention on the knowledge that is significant only to mathematics teachers. This position has given rise to the MTSK model advocated by Carrillo, Climent, Contreras, and Muñoz-Catalán (2012), in which specialisation receives an alternative focus centred on mathematics teachers' specialised knowledge (MTSK), which abandons the

notion of mathematical knowledge for teaching to centre on knowledge of significance only to mathematics teachers.

In the following section we summarise several key aspects of devising descriptors for the six subdomains of the MTSK model. We begin with the three subdomains concerned with mathematical knowledge (MK), which all concern the way teachers' understand mathematics.

Knowledge of Topics (KOT). The fundamental concern of this subdomain is the idea of "knowing" a topic, and this would need to be reflected in the corresponding descriptors. However, it is important to consider everything this knowledge implies, from rules, procedures and calculation methods associated with the concept to the different meanings of a topic in itself (e.g., la derivative as the gradient of a curve or as the limit of finite increments). Equally important is to consider the different phenomena associated with mathematical concepts (Rico, 1997). These are some of the considerations that the group which developed the MTSK model took into account when drawing up the description of KOT. Nevertheless, a thoroughgoing review of the various considerations involved in this subdomain is still necessary. For this subdomain, for the topic of matrices considered in Sosa (2011), we could write a descriptor such as the following, *Knowing why the elements of a specific matrix in a problem are laid out in a particular way*.

Knowledge of the Structure of Mathematics (KSM). In the MKT model, the characterisation of mathematical horizon knowledge tends to make us think of connections as the defining element of this subdomain (drawing on Fernández (2011) and Martínez, Giné, Fernández, Figueiras, and Deulofeu (2011)). However, knowledge of structure represents understanding the connections with elements that are prior and subsequent to the item being studied (Montes, Aguilar, Carrillo, & Muñoz-Catalán, 2012)). It is worth noting that such shifts forward and backwards in time are not so much curricular as mathematical. An example is knowing the connection between the integration and measurement, even though they are not concepts which occur in the same year. Hence, we regard the descriptors in this subdomain as being related to the topic being studied. Thus, one such descriptor might be, *Knowing the relationship between matrix algebra and geometry*, which gives us information about the knowledge of a topic which could be covered in one year, with matrix geometry, which could be from another topic in the same, or other, year.

Knowledge About Mathematics (KAM). This subdomain, which corresponds to the idea developed by Ball (1990), refers to ways of dealing with mathematics (Carrillo et al., 2012; Montes et al., 2012)). In order to provide a description of knowledge about mathematics it is necessary to give preeminence to aspects of mathematical reasoning. For example, the descriptors should incorporate concepts such as definition, demonstration and argumentation, which reflect understanding of what constitutes a definition, or when a demonstration has been completed, or when a particular line of reasoning is valid. Depending on the topic in hand, this subdomain will reflect certain aspects of mathematical procedure, but these will not be concepts inherent in the concept itself. Hence, for multiplying matrices again, a KAM descriptor such as *Knowing that definitions and properties have limits* could be phrased as *Knowing why any two matrices cannot be multiplied*, which would already form part of KOT.

In the following section we will consider the subdomains corresponding to pedagogical content knowledge.

Knowledge of Features of Learning Mathematics (KFLM). It is worth reminding ourselves that understanding the manner in which pupils learn mathematics is not in itself purely mathematical, although it clearly involves mathematics, but it is very closely linked to the work of teaching. In our opinion, teachers' understanding of their pupils' learning is influenced by they way they understand learning, and hence there will be differences between those teachers who endeavour that mathematics be learnt mechanically or meaningfully (Skemp, 1978), and those who do not, or who consider only one of the modes. Thus, the descriptors for this subdomain will totally depend on the topic under consideration, as the learning related to fractions, for example, need not necessarily share the same features as the limit, although theories such as those advanced by Sfard (1991), APOS (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996), or any other theory of learning could prove useful in devising descriptors for every learning state. Such a descriptor from this subdomain, referring once again to the multiplication of matrices, could be phrased as *Knowing that students tend to use the commutative property for multiplying matrices*.

Knowledge of Mathematics Teaching (KMT). The knowledge described in this subdomain is that which allows teachers to take the complex series of decisions that constitutes the task of teaching, such as making the choice of an appropriate textbook, selecting a representation for a particular concept, or finding specific resource material for dealing with a topic. Hence, this subdomain requires descriptors which reflect teachers' decision making

processes for carrying out a lesson. A typical descriptor would be, *Initiates the teaching of matrix algebra using non-square matrices of limited size (as with the scalar product of vectors)*, which draws on knowledge that is considered in other subdomains, but which also has its own existence.

Knowledge of Mathematics Learning Standards (KMLS). In this case, we consider the descriptors proposed by Sosa (2011) for curricular knowledge a good starting point, although we think it necessary to add descriptors relating to teachers' institutional context, such as aspects of knowledge deriving from professional associations (such as the NCTM), journals or research groups, beyond the confines of the prescriptions of educational authorities. As such, it would be necessary to explore which of these teachers considered conventional sources of information. For example, *Being aware of professional papers dealing with student problems with matrix algebra* would be accepted as a descriptor as here research literature constitutes a standard source for developing one's educational knowledge.

A FINAL REFLECTION

As suggested above, the MTSK model represents a change of perspective in the MKT model, given that it considers all specialised aspects making up mathematics teachers' knowledge, including both the teaching profession and the object of teaching, in this case, mathematics. The specialisation of mathematics teachers' knowledge means going deeply into the idea of pedagogical knowledge and distancing on self from it, in order to achieve a 'mathematisation' of the model, which goes from considering the specificity of pedagogical knowledge referring to teaching and learning mathematical content, to focusing on the mathematical pedagogical knowledge which defines the profession of mathematics teacher.

Given that the MTSK model is under construction, we think that one of the most important tasks to see through in future studies is a more precise description of the model, creating descriptors for the subdomains which enable us achieve a better understanding of the nature of teachers' knowledge in line with the above. In like fashion, access protocols to the distinct components of teachers' knowledge are needed to thus resolve the controversies surrounding them, as in our opinion certain subdomains, such as mathematical horizon knowledge in the MKT model, despite being generally accepted as a dimension of teachers' knowledge, has limited accessibility for describing what is observed.

NOTES

[1] Each descriptor with its respective acronym and definition can be found in Sosa (2011, pp. 63-70).

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