HEURISTIC STRATEGIES PROSPECTIVE TEACHERS USE IN ANALYSING STUDENTS' WORK

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In this paper I will describe with what specific heuristic strategies two prospective secondary school mathematics teachers, who are students towards the end of their academic education at the university, analyse a student's work and reconstruct possible thinking processes of this student. The research of these two prospective teachers reported in this paper is part of a qualitative study which investigates prospective teachers' behaviour of analysing students' work consisting of students' written answers to mathematical problem-solving tasks. The prospective teachers were questioned in clinical interviews and videotaped while analysing the student's work individually.

Keywords: heuristic strategies, prospective teachers, teacher education, teacher knowledge, analysis of students' work

INTRODUCTION

Examining students' work and their thinking in order to give feedback is an important part of effective teaching. For future teachers it is important to acquire diagnostic competence in order to understand and assess students' answers with the aim to make appropriate pedagogical and didactical decisions (Hußmann et al., 2007). This study draws on the process of analysing students' work. Particularly the following question is addressed: Which heuristic strategies do prospective teachers use in analysing students‘ work consisting of students' written answers to mathematical tasks?

THEORETICAL FRAMEWORK

Over the last couple of years many research projects have addressed teacher knowledge. Ball et al. (2008) discussed what makes Mathematical Knowledge for Teaching (MKT) special. They include among this, finding out what students have done, interpreting students' errors and assessing whether the thinking and approaches are mathematically correct and would work in general.

Examining students' work and thinking should be a part of everyday teaching practice (Borasi et al., 2002). Mathematics teachers "need to know how to [...] analyse students' solutions and explanations." (Hill et al., 2005, p. 372). Hill et al. (2005) include among the work of teaching mathematics, among others, "interpreting students' statements and solutions" (p. 372). The Teaching Principles of the National Council of Teachers of Mathematics (NCTM) Standards include "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (NCTM, 2000, p. 2).

In fundamental ways Schoenfeld (2011) views "teaching as a much more complex problem-solving activity" (p. 3). Especially when students employ unusual strategies of solving a task students' thinking is not always directly obvious and easily comprehensible. There is a need for teachers to use heuristic strategies to reconstruct possible thinking processes with the aim to understand the students' work. In the analysis of students' work possible thinking processes must be reconstructed in hypothetical, empathetic thinking by interpreting the work.

Crespo (2000) found in a study with elementary preservice teachers that they struggle with interpreting students' work. The elementary preservice teachers tend to evaluate the students' work immediately without analysing it carefully. In my study I had chosen prospective secondary school teachers towards the end of their academic education because I am interested in to what extent the academic education of secondary school teachers prepares for mathematical requirements needed in the analysis of students' solving processes.

Schoenfeld (2011) claims "that what people do is a function of their resources (their knowledge, in the context of available material and other resources), goals (the conscious or unconscious aims they are trying to achieve) and orientations (their beliefs, values, biases, dispositions, etc.)." (p. xiv). The attempt in this study is to regard analysing students' work as a function of knowledge and resources, of goals and of orientations. What people do in this study is analysing students' work. In the following I describe which kinds of knowledge could have influence on these analyses. Figure 1 illustrates the conceptual framework.

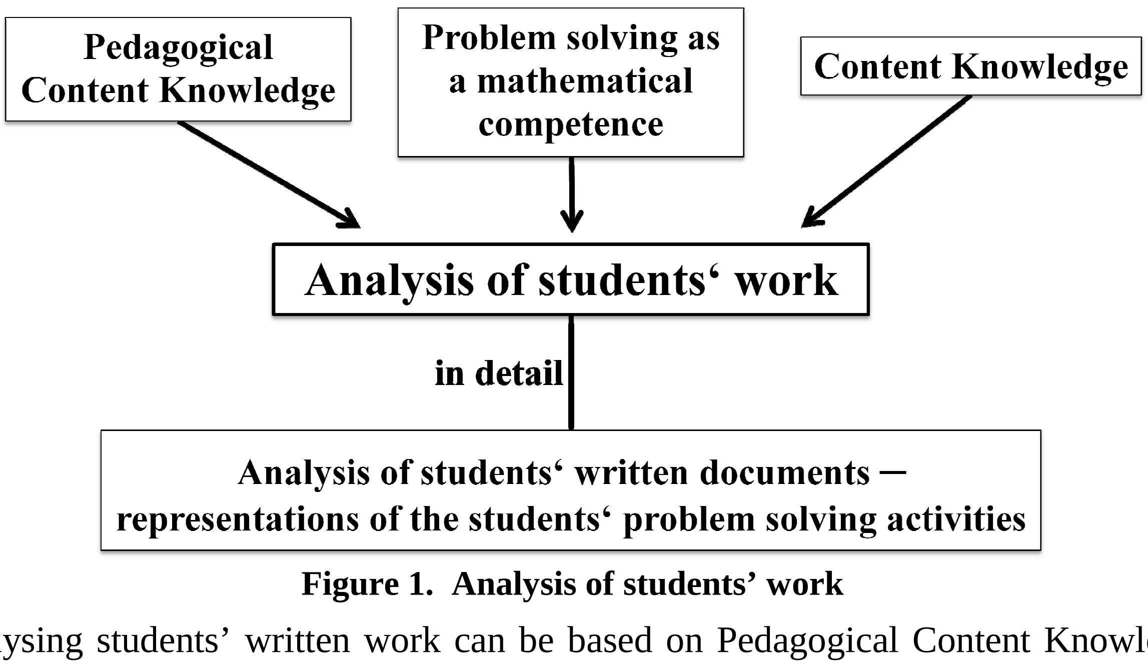


Figure 1. Analysis of students' work

Analysing students' written work can be based on Pedagogical Content Knowledge (PCK), Content Knowledge (CK), and problem solving as mathematical competence. Shulman (1986) includes among pedagogical content knowledge "the ways of representing and formulating the subject that make it comprehensible for others [...] [and] the conceptions and preconceptions that students of different ages and backgrounds bring with them" (Shulman, 1986, p. 9). Content knowledge includes knowledge of the subject and its organising structures (Shulman, 1986). A teacher “need not only understand that something is so; the teacher must further understand why it is so” (Shulman, 1986, p. 9).

In the process of analysing students’ work it is in some cases necessary to change from one representation system to another, for example, if the student has chosen two different representation systems within his or her solution process. Duval (2006) distinguishes two different types of transformations of semiotic representations: treatments (transformation from one semiotic representation to another in the same register) and conversions (transformation from one semiotic representation to another in a different register).

Problem solving as mathematical competence is another important aspect for analysing students’ written work as semiotic representation of the student’s problem-solving activities. Reconstructing possible thinking processes of the student can be based on identifying specific problem-solving strategies in the student’s work. According to Polya (1949) problems are specific tasks in which necessary steps for getting the solution are unknown. Characteristic of problem is that a person has to use a way of solving which is for him or her unfamiliar. The character of problem is not only dependent on the special question or task presentation but also on the knowledge of the person (Heinze, 2007). This is why it is possible that a task may be a problem for some students and it can be a routine exercise with no significant efforts for others (Shulman, 1985). In a transferred sense it is the same with analysing students’ written work in that analysing is not always problem solving. In some cases, especially for experienced teachers, analysing students’ work with the aim to understand their thinking is rather working with presentations than problem solving. (A possible reason could be higher pedagogical content knowledge because of teacher experiences.) For this reason it is interesting to find out which strategies prospective teachers actually use in analysing.

Polya (1945) identifies four basic principles of problem solving: understand the problem, devise a plan, carry out the plan and look back. Typical problem-solving strategies also known as heuristics (Shulman, 1986, p. 26), are forward working, backward working, combined forward and backward working, systematic trying, search for equations, relations or mathematical patterns (Bruder et al., 2011). While solving problems it is possible to make use of heuristic tools like table, informative figures, solution graphs, variables/equations and stored knowledge. (Bruder et al., 2011). Polya (1945) introduces heuristics to describe the strategies or the “mental operations typically useful for the solution of problems” (p. 2). Problem solving requires mental flexibility (Bruder et al., 2011).

METHOD OF INVESTIGATION AND ITS JUSTIFICATION

Research design and sample

The two cases described in this paper are situated within in a qualitative study of 19 prospective secondary school mathematics teachers, towards the end of their academic education at the University of Oldenburg, Germany. In Germany the teacher education is composed of two parts. The first part is an academic education at the university and the second part is a practical training in teacher seminars and in school.

In a two step design prospective teachers were asked to solve three different mathematical tasks, and to analyse one student's written work to each of these tasks. This should be done individually with unlimited time and with no other tools except the help of pencil and paper. The mathematical tasks have multiple ways of solving. Some of the students' written answers are fictive and others are real.

The data includes on the one hand written work of the prospective teachers concerning their answers to the mathematical tasks and their analyses of students' work; on the other hand videotaped, transcribed material of oral comments while solving tasks and analysing students' work and a semi-structured interview afterwards. To understand their ideas in more detail, the prospective teachers were asked to explain their work and their hypotheses on the students' thinking processes in an interview. I had chosen this combination of written and oral form, because to one hand the participants could think more profoundly about different aspects to write down, and they were not in hurry to answer something quickly; on the other hand to get also data about what they perhaps have already thought concerning the student's work but did not write down. Additionally the prospective teachers are working with the student's work twice thereby they can get more ideas.

In this paper I describe the heuristic strategies two prospective teachers (Marc and Tim) show in analysing Lilly's written work to the task "Find the solution set", one of the three different mathematical tasks. I want to describe the analyses of these two participants because they use some identical strategies and some which are completely different from each other. In comparison with the other 17 participants of the whole study, the strategies of these two cases are also used by other participants.

The mathematical task and Lilly's answer are shown in Figure 2.

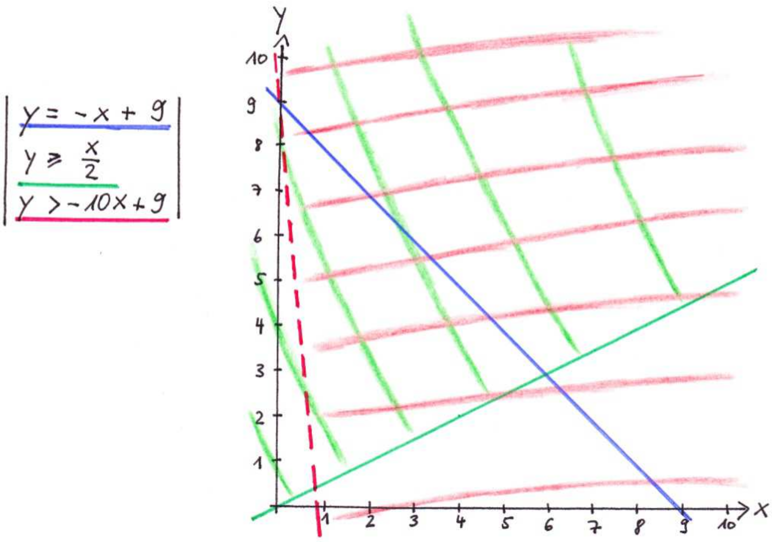
<u>Mathematical problem "Find the solution set"</u>

Find the solution set of $\left|\begin{array}{rcl} x+y & = & 9 \\ x-2y & \leq & 0 \\ 10x+y & > & 9 \end{array}\right|$ *where x and y are natural numbers.*

Describe your approach and document your way of solving as accurately as possible. What alternative approaches do you know? Carry them out.

<u>Lilly's work to this task</u>

Lilly has rearranged the system and created the following illustration in a Cartesian coordinate system.



What thinking process do you conjecture in Lilly's work?

Figure 2. Mathematical problem "Find the solution set" and Lilly's work

For this task content knowledge about the meaning of such a system, the solution set and the concept of variables is needed. Depending on the chosen solution strategy more content knowledge, such as the substitution method or knowledge of geometric illustrations is required. German students are familiar with systems of linear equations but not with dealing with linear inequalities. They have to transfer their problem-solving strategy for solving a system of linear equations to a system with inequalities or to think about another solution strategy where mathematical process skills will be needed.

Lilly has rearranged the inequalities and the equation for y by treatment within the algebraic register. She changes into the graphic register (conversion according to Duval, 2006) with her other representation. In the xy-plane the solution set of each inequality and equation is graphed as lines or areas for x and y ∈ ℝ. A possible reason why the line for y=-10x+9 is drawn as a dashed line could be that the set of the line is not included in the set of y>10x+9. The missing solution set of the complete system can be found graphically as the intersection of the three individual solution sets of the equation and the two inequalities. Since x, y ∈ ℕ, the solution set consists of the following ordered pairs: IL={(1/8),(2/7),(3/6),(4/5),(5/4),(6/3)}.

Data analysis

To find out heuristic strategies the prospective teachers use in analysing Lilly's work I go through the written and transcribed material in combination both line by line for analysing and interpreting significant statements and strategies with the aim to expose categories for heuristic strategies.

ANALYSES OF LILLY'S WORK

In the following I want to describe the strategies that two prospective teachers (the case of Marc and the case of Tim) use to analyse Lilly's work. The prospective teachers had unlimited time to analyse Lilly's work, both of them worked on it 10 minutes. Before analysing the student's work the prospective teachers were asked to solve the mathematical task on their own.

Marc

Marc chooses in his own solution the substitution method (a target-aimed method for this task). He rearranged the equation for x and replace it in the inequalities. His way of solving indicates the safe use of algebraic transformations and solving inequalities. In his solution process, he shows strength in formal-operational activities. Marc finds out two correct restrictions for x and for y (namely $0<x\leq6$, $3\leq y<9$). However, he refers to the restrictions found for the variables separately but not combined as ordered pairs and without regard to the condition $x+y=9$. Marc claims that he cannot write down a solution set.

Marc explains what thinking process he conjectures in Lilly's work after a few minutes looking at her written work and writing down some aspects. Marc mentions:

> Marc: She sees in y this function and tries to write down every equation, which is here [points at the system with equation and inequalities], as a function and tries to solve it with a drawing and yes, she has done it right. […]
>
> She is now thinking the solution set is everything which is on the line. At this it is all what is over it and for y3 it has to be definitely bigger, therefore she has drawn it as a dashed line, because it doesn't include this value thus this line.

Lilly has rearranged the inequalities and the equation for y. Hence he concludes that she will regard it as a function. A possible reason for this could be that her rearranged conditions remind him of the standard presentation of functions. Before commenting this orally he wrote down

- Lilly sees in y the function f(x), rearranged the equations for y
- equation=function

This indicates that he sets himself a framework namely the context function as a fundamental idea concerning Lilly's work. This fundamental idea of function occurs also at the end of his analysis.

In his written and in his oral analysis he numbers consecutively the three conditions for y in the algebraic representation of Lilly's work (y3 for the third condition). Hereby he separates the conditions for y from each other (deliberately or not).

Marc tries to reconstruct the action by splitting up Lilly's work and analysing it separately. He splits up the solution in two ways: 1) At the beginning of his analysis he thinks in the algebraic register and changes into the graphic register (Duval, 2006). 2) Marc interprets the equation and the inequalities separately as straight lines and as designated areas in Lilly's graph.

He interprets elements of the solution locally like the graphed solution set of the equation and each inequality as lines or areas. In his analysis of Lilly's work he is influenced by his attitude towards the correctness of the work. Some of Marc's statements indicate that he thinks something is wrong but he is unsure. Nevertheless he takes it seriously and tries to understand Lilly’s approach.

Marc: What would happen now, if I take a value for x, which is smaller than one? […] She need not to regard this, it is completely indifferent for her.

In this excerpt his analysis goes on with using a “what if”-strategy as occasion to think from Lilly’s perspective. This approach does not advance him in reconstructing the thinking processes. He rejects his strategy because he thinks that it is irrelevant. Later he thinks about how Lilly could go on working on this or how he would help her exactly.

Marc: Perhaps one can go on working on this but I don’t know it now. […] What would be, if I interpret it with my solution? So, I don’t know, how do it with that but if I take my solution to interpret it in connection with this solution.

Following this he tries to deduce the thinking process in Lilly’s work with the help of his own solution. He shows his conditions ($0<x\leq6$, $3\leq y<9$) in Lilly’s work at the corresponding axes, but he does not respond to a combination of x-and y-values with respect to the solution set. Marc finds his conditions in the graphic representation but he does not understand Lilly’s representation. It can be said, that he does not succeed in understanding how the graphic representation of Lilly is related to the determination of the solution set of an algebraic system. His main problem is the mathematics. In the analysis of Lilly’s work, he shows a weakness in the cross-linking by not associating his algebraic solving method with the graphic determination of the solution set of an algebraic system. So, he cannot complete Lilly’s solution idea or reconstruct a possible thinking process completely even with the help of his own solution.

Marc can explain details but he cannot relate the three conditions (equation and inequalities) in Lilly’s work because he describes three solution sets. It shows that this is an analogical problem as in his own solution because he does not relate his two found conditions. Altogether he does not grasp the complete strategy of Lilly’s approach but rather the dealing with the separated conditions. He interprets some aspects of the solution locally.

Summary of Marc’s strategies:

- reconstructs the action by splitting up the work and analysing it separately
- deals with representations
- “what if”-strategy as occasion to think from Lilly’s perspective
- uses his own solution to interpret her work

Tim

In his own solution, Tim makes use of the substitution method by rearranging the equation for x and replace it in inequalities. Tim exploits two correct conditions for y ($y\geq3$, $y<9$). He writes down a solution set for y ($y=\{3,4,5,6,7,8\}$) and for x (x={1,2,3,4,5,6}) in a way that x and y are separated. He does not write down ordered pairs (x, y) regarding the condition x+y=9.

By analysing Lilly's work, Tim mentions:

> Tim: First she tried to put the whole thing down to something familiar namely to a linear system of equation and then look at each as intersection of two lines.

When Tim describes Lilly's approach, he identifies Lilly's strategy which I call "drawing on something familiar".

> Tim: This matches. This intersection point [points at P(6/3) in Lilly's graph] x equals 6 is ok and also y which is 3. Thus the points [points at his solution sets] are included.

Tim refers to intersection points. He checks if the intersection point (6/3) that he looked at in Lilly's graph is included in his own solution because he intends to verify Lilly's work. Then he states that his solution and Lilly's graphic representation is consistent. This excerpt is an indicator for the strategy "referring to his own solution".

He says that Lilly did not mark the solution set and that he does not quite see it. Following this he mentions:

> Tim: She didn't understand the question of the task which is to find the solution set [points at the task] which fulfills this. This she didn't do exactly because […].

By referring to Lilly's work Tim points out that Lilly's solution is not completed, when he explains that Lilly did not answer the task. He assumes that Lilly did not understand the question and takes the graphic representation into account. He points at the point (0/9) and says that this would be possible in this representation but this does not fit to the fact 9>y (he points in the direction of the task), namely that it is only included up to the point (1/8) at which he is pointing. It indicates that he examines the correctness of the solution like he has done at the beginning of his analysis. Possibly Tim refers to his own solution again (however implicit) because he knows that the point (0/9) is not included in the solution set.

The same as in the case of Marc Tim applies the strategy of splitting up the solution. Thereby he focuses on single transformations and changes from algebraic register into graphic register. Later he mentions that something special is the dependence in the solution set. Take it together it has to be nine in each case.

Summary of Tim's strategies:

- identifies Lilly's strategy "drawing on something familiar"
- reconstructs the action by splitting up the work and analysing it separately
- deals with representations
- verifies Lilly's solution by referring to his own solution

Comparison of the two cases

Marc and Tim use some identical strategies to analyse Lilly's work (reconstruct the action by splitting up the work and analysing it separately; deal with representations). Both of them refer to their own solutions but in a different way. Marc uses his own solution to interpret Lilly's work whereas Tim refers to his solution to verify special aspects of Lilly's work.

Marc has difficulties in understanding Lilly's work. Marc thinks something is wrong but he is unsure, so he takes it seriously and tries to understand Lilly's approach. He interprets details of her work locally but cannot reconstruct a possible thinking process completely even with the help of his solution. Tim gives greater consideration to Lilly's strategy which he identifies in her work. Altogether he gives the impression that he rather verifies Lilly's work.

CONCLUSIONS

Concerning the answer of the research question the analyses of the two prospective teachers were analysed to find out their heuristic strategies by which they analyse a student's work and reconstruct possible thinking processes. The results of these cases show that analysing a student's work evokes specific heuristic strategies and that different heuristic strategies could be reconstructed. Marc and Tim used similar and different strategies in their analyses. Both of them used the strategy "referring to his own solution" which seems to be a typical strategy. "Dealing with representations" as another heuristic strategy is one which is used frequently with the aim of understanding students' work, especially if the students employ unusual strategies of solving a task.

With the help of the whole study of all participants further strategies can be investigated. So far, the analyses of the data show that some prospective teachers have difficulties in analysing students' work and tend to evaluate the students' work immediately without analysing it in a deeper way. Other prospective teachers try to reconstruct the student's thinking processes carefully and do not only describe obvious activities of the student, e.g. describe not only written calculus but also possible ideas of the students' solution strategy. These differences which may be caused in different orientations, strategies or differences in available knowledge need to be investigated. This can make contribution to the development of the academic teacher education by making proposals e.g. of analyzing students' work what can be regarded in a didactic course at the university. Some prospective teachers can improve their analyses by knowing further strategies to be more flexible in their analyses of student's work. According Maher and Davis (1990) "[p]aying attention to the mathematical thinking of students engaged in active mathematical constructions, and trying to make sense of what students are doing and why they are doing it, is prerequisite" (p. 89). This is an important part of effective teaching.

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