MATHEMATICAL COMMUNICATION: TEACHERS' RECOGNITION OF THE SINGULARITY OF STUDENTS' KNOWLEDGE

António Guerreiro[2], João Pedro da Ponte[3] and Lurdes Serrazina[4]

[2]School of Education and Communication, University of the Algarve (Portugal)

[3]Institute of Education, University of Lisbon (Portugal)

[4]School of Education, Polytechnic Institute of Lisbon (Portugal)

This paper discusses the role of collaborative work in fostering social interactions in the classroom, how teachers value such interactions to develop the modes of communication, and in the interaction patterns centered on students' individual knowledge. Data were collected in the context of collaborative work involving a researcher and three participating teachers. The development of interactions among the students themselves and between them and the teachers, along with the recognition of the students' singular mathematical knowledge, led to the adoption of reflexive and instructive modes of communication and also to the extraction and discussion patterns in mathematical communication.

Key words: mathematical communication; modes of communication; interaction patterns; teaching practices; collaborative work.

INTRODUCTION

Mathematical communication is as much part of mathematics classes as mathematics itself. Such communication takes different forms according to the teachers' conceptions of the nature of mathematics, its teaching and learning and the role of communication in their professional practices. Teachers' positions concerning the nature of mathematics as a way of understanding society are translated into forms of mathematical communication, considering the existence of several strategies and singular forms of mathematical knowledge in the classroom.

Mathematical communication as the foundation of the teaching and learning process is based on the recognition of social interactions in the classroom between the teacher, the students and mathematical knowledge (Sierpinska, 1998) and considers discourse as a social practice (Godino & Llinares, 2000), where the role played by all actors in the negotiation of mathematical meanings is of the utmost importance. In this perspective, mathematical communication is a social process in which participants interact, exchanging information, influencing each other, taking up the attitude of the other and, simultaneously, expressing and asserting his or her singularity (Belchior, 2003; Mead, 1992). Teaching becomes the organization of an interactive and reflexive process, where the teacher is continuously engaged in updated and differentiated activities with students (Cruz & Martinón, 1998) and where students become aware of their cognitive and affective processes, mobilizing them for learning (Ponte, Brocardo & Oliveira, 2003).

This paper looks into in the teachers' recognition of the interactions among the students themselves and between them and the teacher, as well as the way in which these social interactions are valued in recognizing the singularity of the students' mathematical knowledge, in the context of a collaborative work. We focus on the relation between the development of social interactions among students, between and them and the teacher, and the recognition of students' mathematical knowledge. In a specific way, we question how the valorization of the students' mathematical knowledge is related to the modes of communication and to the interaction patterns centered on reflecting and sharing of knowledge among all the actors in the mathematics classroom.

MATHEMATICAL COMMUNICATION CLASSROOM PRACTICES

Mathematical communication classroom practices are structured into several modes, in accordance with the teacher and the students' role in classroom discourse. Brendefur and Frykholm (2000) classified them as unidirectional, contributive, reflexive and instructive communication. The unidirectional and contributive modes of communication are associated with the teacher's absolute, or almost absolute, power over the classroom discourse. Reflexive and instructive modes of communication show the importance of classroom discourse as an object of reflection and sharing of knowledge between the teacher and the students, which takes on a metacognitive and reflexive dimension.

The nature of the interactions between teacher and students is expressed by interaction patterns that swing between the control of the students' thought on behalf of the teacher and the sharing of mathematical ideas and strategies through reflexive discussions about the processes of construction of mathematical knowledge. The existence of reciting, funneling and focusing patterns (Godino & Llinares, 2000; Wood, 1994, 1998) results from the centering the students' thought and knowledge on the teacher's knowledge by means of routine processes of reproduction of mathematical knowledge. Extraction and discussion patterns (Godino & Llinares, 2000; Wood, 1994, 1998) show the students' individual contribution in constructing mathematical knowledge, through a collaborative dialogue with the teacher.

RESEARCH DESIGN, ACTORS AND COLLABORATIVE WORK

Access to empirical data concerning classroom practices, conjugating action and meaning, led to adopting the case study design (Stake, 1994) for this study, relying on participant observation and inquiry, with the development of collaborative work, grounded on the reflection and questioning about communication practices in the classroom. Three teachers of the 1st cycle of basic education − Alexandra, Carolina e Laura (pseudonyms) – took part in this study for a period of over two years (December 2006 – February 2009), in the course of which there were 12 meetings of collaborative work (collaborative meetings) (audio recorded), structured around the teachers' communication practices in the classroom, 14 classroom observations (video and audio recorded) by each teacher (participant observation) and two semistructured interviews and two individual meetings (audio recorded) by each teacher (inquiry) (for more details see Guerreiro, 2011). These interviews and individual meetings contributed to understand the teachers' meanings about changes in interactions with students and in their classroom professional practices.

These teachers had a significant professional experience, with a minimum of 12 years of teaching in 1st cycle (Alexandra and Carolina, 12 years, and Laura, 17 years), and were highly motivated to work in collaboration with a view to strengthen their professional knowledge about mathematical communication in the classroom. Before this study, the teachers attended an in-service training course in mathematics where mathematical, didactical and curricular knowledge were addressed. The definition of common interests and goals resulted from the need to ensure a direction for the collaborative work with regard to the teachers' professional development (Boavida & Ponte, 2002) and to the construction and reconstruction of classroom practices, giving way to a reflective constancy and successful changes in the partnership that existed between the first author of this paper (from now on referred to as "the researcher"), and the participating teachers (Ruthven & Goodchild, 2008).

The axes of collaborative work were trust, as referred to by Boavida and Ponte (2002), and the valorization of the critical analysis of the researcher and his colleagues, who acted as external observers, as mentioned by Saraiva and Ponte (2005). However, the teachers' initial response after the first classes were conducted and discussed was of anxiety in relation to the quality of their performance and that of their students', evidencing insecurity about their knowledge and professional performance. The development of critical classroom practices and progressive sharing of perspectives among teachers and the researcher (the first author) in the collaborative work went on and was replaced by a significant level of comfort and complicity among participants.

Self-knowledge about teaching practices through the viewing of classes and critical reflection fostered by the researcher led to changes that were acknowledged by teachers and within the scope of the students' group work and were visible in the presentation of mathematical tasks, in accepting errors as educational resources, in the promotion of communicative interactions between the students themselves and between them and the teacher, as well as the recognition of the students' personal and individual mathematical ideas and strategies.

This attitude towards the mathematical ideas of the students and of others justified the teachers' growing disposition to further their training in order to expand their professional and mathematical knowledge. The changes that the teachers made resulted from the recognition of the students' singular knowledge. They also indicate an effective valorization of learning as teachers look upon the students as autonomous learners within the context of communicative sharing marked by the negotiation of mathematical meanings.

RESULTS

Modes of communication

The usual communication practices of the teachers seem to result from the conception of the roles of both teacher and student as doers of structured and finished mathematics, based on solving mathematical tasks. In this perspective, mathematical communication is characterized by the dominant role of the teacher in the classroom discourse as the holder of knowledge to be passed down and the role of the student as keen listener. This is what defines unidirectional and contributive modes of mathematical communication, through the inclusion of examples, solution strategies and correct solutions of exercises and mathematics problems.

The teachers in this study felt relatively uncomfortable in countering these practices of educational and communicative control, and were unable to conceal a significant effort not to condition the autonomous work of the students and to accept alternative or diverging solutions, giving some scope for the comparison and sharing of different solutions of mathematical tasks, even if confusing, incomplete or incorrect. The growing participation of students in classroom discourse, through the valorization of the moments of communicative interaction within the group or the class, led to an important autonomy on the part of the students and to teachers' reflection in action, incorporating in the discourse those ideas and difficulties expressed by the students or felt by the teachers, thus promoting reflexive and instructive modes of communication:

> [We have to] give them time and tasks where they interact with each other; to give them time to interact, to understand, to discuss strategies, first within the group, without having to keep an eye on them all the time, guiding them, [or] leading them to the right answer; to give them time and some room.
>
> [February 2009 _ interview _ Alexandra]

The way in which the teachers learned to increasingly value the personal knowledge of each student resulted in a significant reinforcement of the sharing of knowledge among all actors and in the recognition of the value of the knowledge held by the students and teachers alike. Such a change increased the students' critical awareness and sense of responsibility in the autonomy with which they validated the solutions of mathematical tasks, as a feature of reflexive communication. This strengthened the sharing of mathematical ideas and strategies, instead of explicating the solution of tasks.

The instructive dimension of communication derived from the debates about the reasonability of the results, from the reformulation of incorrect solutions, from the process of knowledge construction generated by solution of the mathematical tasks and from the recognition of the students' learning. Carolina's students began to compare their strategies to reach a mathematical solution regularly, thus showing a greater autonomy in the comprehension of alternative processes:

Jessica (reading the question paper): − Antonio needed a fence seventy-six meters long to enclose his backyard… (she does not finish reading the question)

Dennis: − We wrote a division statement where we divided the figure by two (he writes the traditional division algorithm on the blackboard): seventy six divided by two, to see how much is half of it (the student makes the calculation resorting to his notes). And then we divided thirty-eight by two to estimate the half of thirty-eight to get the answer (he makes a new calculation). And then to see if it works out correctly we (he consults his notes) calculated nineteen times four (he writes an addition sentence with equal parcels). And it makes seventy-six. Therefore, our equation is correct.

Beatriz: − Why didn't you just divide it by four?

Teacher: − Your colleague is asking you a question!

Dennis: − I wanted… wanted half of this.

Beatriz: − Half? But if the square has four sides, I think that instead of dividing seventy-six by two, well, you could simply divide it by four, since the square has four sides.

Dennis: − We did it this way...

Beatriz: − You did it step by step?

Dennis: − We divided it by two, and then, after getting those two, we would find out the other two.

Beatriz: − Yes, I know. You did it step by step, didn't you?

Dennis: − Yes.

Beatriz: − It's just that you could have done it all in one. Instead of thirty-eight divided by two, you could have gone straight on to divide it by four. But OK, you did it step by step, which is all right, too.

Dennis: − More questions?

[March 2008 _ class _ 3rd year _ Carolina]

These occurrences are characterized by a metacognitive dimension of reflection about the action itself, assigning value and integrating the students' knowledge. The teachers' assumption of the specific and personal knowledge of each student gave birth to the development of a reflexive and instructive communication, as well as the increase in the students' autonomy in the teaching and learning process, besides the students' recognition of the mathematical knowledge of the other students.

Interaction patterns

The permanent validation of the students' solution in the course of such an autonomous work seems to result from the uniformity of strategies and solutions expunged of errors and fitting the teachers' knowledge. This attitude on the part of the teachers derives from a learning process centered on the solving of mathematical tasks, without imprecisions or ambiguities, within a short time span. This drives towards the right answer avoiding errors owes much to the initial interaction practices between teachers and students according to the reciting, funneling and focusing interaction patterns based on the teacher's knowledge:

> Sometimes, as I have my own strategy, when they divert from it and are confused, maybe, and have some difficulty in explaining how they managed it, when I explain, I explain it my way. (…) when they are moving away from it, instead of listening through to the end, we cut it short to guide them. (…) Maybe not to waste time and not to let them create wrong concepts.
>
> [January 2008 _ meeting _ Alexandra]

Accepting and understanding the error as a learning resource, conjugated with the valorization of the interactions between students and the decentering of the teachers in relation to the blackboard in the classroom, resulted in the valorization of the students' individual and personal knowledge, and triggered the appearance of extraction and discussion patterns based on the recognition of the students' mathematical thinking. The existence of these patterns reveals the valorization of the students' individual thinking on the part of the learning community.

The extraction pattern results from the valorization of solutions that were different and potentially incomprehensible to other students and the teacher herself, giving way to a questioning pattern where one intends to recognize the validity of ideas, solving strategies or solutions presented by students. In the above problem that consisted in determining the length of the side of a square from its perimeter, one group of students chose to go on dividing by two successively in order to establish the fourth part of the initial figure, as illustrated above. Carolina tries to understand the boy's thinking using and extraction pattern:

> Dennis: − Does anyone else want to ask a question? Yes, please?
>
> Teacher: − I do. I heard what you explained to Bia. Bia ended up explaining, ended up saying, getting your explanation, but I want to know the reason why it crossed your minds to divide seventy-six by two.
>
> Dennis: − To find out how much the half was.
>
> Teacher: − One question: the half of what?
>
> Dennis: − Half of seventy-six.
>
> Teacher: − Why?
>
> Dennis: − To find out the half...

Teacher: − And that half of seventy-six is what? What is it half of?

Dennis: − Where, Ms.?

Teacher: − Mind you, I'm not saying it is wrong, Dennis. Did I tell it was wrong?

Dennis: − No.

Teacher: − OK. I just want to know… to figure out… to try to understand what you were thinking.

[March 2008 _ class _ 3rd year _ Carolina]

Questioning seems inconclusive, which warns us about the difficulty to know the singularity of the knowledge of the other, even through a questioning based on an extractive patterns. Such difficulty seems to have roused a moment of uncertainty in the student in relation to the correction of his solution, which made the negotiation of mathematical meanings problematic. Likewise, the discussion pattern manifests itself in the way the teacher helps to publicize and explain the different types of reasoning, solving strategies or solutions present by the students in group/class. Laura contributes to the explanation of the students through a set of questions so as to guarantee that the students' solutions are validated and accepted by the whole class. In the next classroom episode, the algorithmic solution of an addition statement containing many parcels was set against a strategy of mental calculation involving notes to support it.

Teresa: – We did the following: we added six and five, which is B plus E, six plus five, which makes eleven. And then we added four and four, which makes eight; this makes nineteen.

Miguel: – We started with the larger numbers.

Teresa: – Then we added four, which makes twenty-three. And then three plus three, this makes six, twenty-three, twenty-nine, and then one. This makes thirty.

Teacher: – Doubts?

Students: – No.

[November 2008 _ class _ 3rd year _ Laura]

Faced with different solutions, Laura takes on the role of inquirer aiming to clarify procedures, conjugating extraction of knowledge with the sharing and discussion of solving strategies among the students:

Teacher: – So how did you proceed so as to avoid getting lost?

Miguel: – We ticked the numbers that he had already added.

Teresa: – We added E and B, which makes eleven, and we drew a dotted line, an X to know (she writes on the board as she speaks).

Miguel: – Right.

Teresa: – We then added G, F and C (she writes on the board as she speaks).

Teacher: – Because those are worth what?

Teresa: – Four.

Teacher: – Each.

Miguel: – And we started with the large ones.

Teresa: – We added G and F, that makes eight, plus… (she follows the letters with the piece of chalk) twelve (she writes "twelve").

Teacher: – And now how much do you have?

Teresa: – We already have twenty-three. Then we went on to add H and A. They are three each, it makes six. Twenty-three plus six makes twenty-nine (she writes the partial results). And then we drew another dotted line and added another one …

Teacher: – Which was the line that was missing.

Teresa: – And we got thirty in the end…

Miguel: – … of the news stories.

[November 2008 _ class _ 3rd year]

The revelation of these patterns results from the teachers accepting the existence of singular mathematical strategies on the part of the students, instead of the normalization of mathematical knowledge grounded on the teacher's knowledge. The teachers' recognition that it is possible to conduct a positive exploitation of the error and that students have a personal knowledge of their own resulted in a sharing, among all, of the singular knowledge of every student, based on patterns of discussion and extraction.

MATHEMATICAL COMMUNICATION AS VALORIZATION OF STUDENTS' KNOWLEDGE

The collaborative work took on a significant role in the development of teachers' knowledge of their mathematical communication practices. These were broadened by viewing classroom episodes and by the critical questioning of the researcher, particularly by recognizing the existence of a practical communication focusing on teachers' knowledge, originated by the anticipation and validation of students' mathematical strategies, oriented for the standard solution of mathematical tasks. This collaboration resulted in a change in teaching practices based on the valorization of the interactions between students and these and the teacher, in the recognition of errors as a learning resources and in the recognition of the singularity of the students' ideas, strategies and mathematical knowledge. Alexandra minimized the intervention of anticipation of students' mathematical strategies to appreciate the interactions between them and recognize the students' correct, confusing and incorrect mathematical ideas. Carolina has developed a significant autonomy in her students, encouraging her students to contrast different resolutions and notice singular mathematical ideas. Laura developed the students' recognition of the error as a way of learning to appreciate the importance of each student's mathematical ideas. The progressive valorization of the individual students' mathematical knowledge induced a change in mathematical classroom communication practices, encouraging the teachers to broaden their mathematical knowledge, as a way to contribute to the understanding of students' mathematical knowledge.

The path from the initial mathematical communication practices, grounded on teachers' knowledge, and the later mathematical communication practices was largely based on the valorization by teachers of the interactions among students and between students and teacher as well as of the singularity of students' knowledge. This led to the strengthening of the students' capacity to communicate their mathematical ideas and to interpret and understand the ideas of the others. The deepening of social interactions in the classroom promoted the existence of reflexive and instructive modes of communication, generating a sharing of responsibilities between the teacher and the students in the recognition of the mathematical knowledge constructed in the classroom. The teacher's acceptance of the students' original mathematical ideas and strategies, though confusing, incomplete or incorrect as a valid learning resource triggered processes of knowledge comparison fed by discussion and extraction patterns centered on students' knowledge.

In this way, mathematical communication practices went far beyond the reproduction of the teachers' mathematical knowledge and assumed an intentional action in valuing the singular mathematical knowledge constructed by each student. The existence of reflexive and instructive modes of communication and of patterns of extraction and discussion evidenced students' mathematical knowledge, generated a significant level of autonomy in the construction and validation of mathematical solutions. The recognition of the existence of diverse mathematical ideas and strategies on the part of students and teachers led teachers to assume a respect for the opinions, mathematical ideas and strategies of students.

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