EXPLORING THE POTENTIAL OF COMPUTER ENVIRONMENTS FOR THE TEACHING AND LEARNING OF FUNCTIONS: A DOUBLE ANALYSIS FROM TWO TRADITIONS OF RESEARCH

Jean-Baptiste Lagrange — University Paris Diderot, France

Giorgos Psycharis — University of Athens, Greece

In this paper we aim to address the potential of computational environments offering integrated geometrical and algebraic representations for the teaching and learning of functions. We follow a 'double analysis' method to analyse learning situations of an experiment that took place in the French context through the lens of the original research tradition (Theory of Didactical Situations) and an 'alien' one (Constructionism). The analysis indicates that this method enhances our efficiency to capture aspects of research traditions which influence knowledge concerning the nature of learning situations for functions with computers.

INTRODUCTION

The notion of function occupies a central role in a wide range of mathematical topics, but engaging students in functional thinking is known as a demanding task. We note issues identified by research in relation to students' difficulties in understanding function as covariation (Carlson et al., 2002) and dealing with algebraic symbolism (e.g. distinguishing between independent dependent and variables, Thompson, 1994). The development of new modes of representation within specially designed technological tools has generated further interest as regards their potential to deal with the above mentioned difficulties. One of the prime affordances of such tools is the multiple linked representations designed with the aim of providing some sort of combination of Dynamic Geometry Systems (DGS) and algebraic multirepresentation, possibly including Computer Algebra Systems (CAS) (Mackrell, 2011). In this study we are especially sensible to the possibility offered by particular computational environments to connect the notion of function to dependencies and covariations between geometrical objects. Existing research indicates that geometrical situations in DGS can be a fruitful context to challenge students' intuitions and ideas about covariation and functional dependency come into play (Falcade et al., 2007). In addition, problem solving by way of algebraic modeling of geometrical dependencies, which includes also sensual experience of these dependencies, can provide a basis for students' understanding of the idea of function provided that students can work flexibly between the geometrical and symbolic settings (Lagrange & Minh, 2010). Here we report research aiming at shedding light on the potential of computational environments offering interconnected algebraic and geometrical representations to facilitate students' making of links between the qualitative experience of dependencies in the geometrical context and the algebraic notion of function. Yet, it seems difficult to really appreciate this potential, since it is needed to take into account the visions provided by specific theoretical frameworks in technology enhanced mathematics, and because of the fragmented character of these frameworks (Artigue, 2009). Here we consider fragmentation as resulting from the existence of different research traditions. By research tradition, we do not mean only a reference to a theoretical framework, but also all the research practice built jointly with a framework: reflection on a practice gives theoretical elements for a framework, and, in return, practice, constituted by design, observation and interpretation, is affected by the framework. While compartmentalized research traditions participate into fragmentation, we assume that different research traditions can be confronted and in some sense articulated in order to address a particular question, the potential of computer environments for the teaching and learning of functions. Testing this hypothesis is the general aim of this study.

Our choice here is to consider two research traditions, both dealing with functions and software, but different in many other aspects. One involves *Casyopée*, a piece of software that offers a dynamic geometry window connected to a symbolic environment specifically designed to support students' work on functions (Lagrange, 2005). Casyopée's design and experimentations occurred in a French context shaped by didactical theoretical frameworks and epistemological considerations. We focus here on a framework preeminent in the French context: the Theory of Didactical Situations (TDS) (Brousseau, 1997). The other research tradition involves *Turtleworlds*, a piece of geometrical construction software which combines symbolic notation (Logo) with dynamic manipulation of variable values (Kynigos, 2004). The design and the research on the use of Turtleworlds is inspired by Constructionism (Papert, 1980) and have been carried out in the Greek context. Constructionism and TDS share a common focus on the design of learning situations through devices – such as the "milieu"- providing affordances for interaction and knowledge construction. Our assumption is that divergent views of the two traditions on the contribution of milieu (e.g. the design and analysis of a session, the nature of the constructed knowledge and its relation to the official knowledge) provide a complementary way to address the potential of computational environments for the teaching and learning of functions.

We drew on data from two concrete teaching experiments taking place in France and Greece respectively. We consider here the teaching experiment designed and implemented with Casyopée in the French context. We carry out "double analysis" of this experiment by way of TDS (a priori and a posteriori analyses) and Constructionism in order to be able to tackle under an "integrated" perspective (in the sense of Prediger et al., 2008) the following question: what new insight about the potential of computer environments offering integrated geometrical and algebraic representations for the teaching and learning of function might be gained from the double analysis of research studies carried out by researchers following different research traditions in different national and didactic contexts? Prior joint research experience in cross-analysing teaching experiments with use of digital tools for function (project ReMath [1]) revealed that misunderstanding rooted in divergent views of functions and distinctive theoretical orientations could be addressed at two levels: one is the economy of learning situations, to which we refer in this paper, and the other is the process of conceptualisation of functions by students. Thus, the links between the issues involved in our research focus here are as follows: investigating the potential of computational environments for the teaching and learning of functions raises the issue of fragmentation of theoretical frameworks in the field of technology enhanced mathematics; in order to explore such fragmentation at the level of research traditions we adopt methods (such as double analysis) and theoretical tools (such as the economy of learning situations) that promote a deeper focus on the design, implementation and analysis of teaching experiments taking place in different contexts.

THE ECONOMY OF LEARNING SITUATIONS

The notion of “economy” of learning situations helps to address the role of the many different components intervening in the classroom progression of knowledge: students, teacher, but also various artefacts which can be material (e.g. blackboard, disposition of the room) or not (e.g. tasks, rules, systems of notation, language). According to Hoyles, Lagrange and Noss (2006, p. 301): “… a learning situation has an economy, that is a specific organization of classroom components, and technology brings changes and specificities in this economy …”.

Theory of Didactical Situations (TDS)

Brousseau (1997) presents TDS as a way to model mathematical situations in a learning context. In this model, a central notion is the “milieu”, a device which justifies the use of knowledge objectively to solve a given problem. Student’s acting on the “milieu” provokes feedback calling for modifying or adjusting action. Learning thus results from the student’s adaptation to an antagonist “milieu”. Teaching consists in organising these constraints and keeping optimal the conditions of the interaction. TDS considers adidactical situations designed in a way that the desired outcome can be obtained only by applying the knowledge aimed at in the situation. Researchers who refer to TDS in order to consider interactions with digital environments (e.g. Cerulli et al., 2008) propose to think of technological learning environments as means to provide students with an antagonistic milieu, offering tasks and feedbacks adequate for the knowledge at stake, under the condition that situations of use are based on a suitable a priori analysis.

Constructionism

Constructionism incorporates and builds upon constructivism's connotation of learning as "building knowledge structures" through progressive internalization of actions, in a context where students are consciously engaged in constructing (or de/re-constructing) physical and virtual models of situations on the computer (e.g. geometrical figures, simulations, animations): the notion of construction refers both to the ‘external’ product of students’ activity as well as to the process by which students come to develop more formal understandings of ideas and relationships (Papert, 1980). The constructionist paradigm attributes special emphasis on students’ construction of meanings when using mathematics to construct their own models during individual and collective 'bricolage' with digital artefacts, i.e. continual reshaping of digital artefacts by the students in order to complete specific tasks.

CASYOPÉE

Casyopée deals with various representations of functions. It provides a symbolic window (Fig. 1, left) with tools to work with functions in the three registers: numeric, graphic and symbolic. Casyopée also includes a dynamic geometry window (Fig. 2, right) linked to the symbolic window. The geometric window allows defining independent magnitudes (implying free points) and also dependant ones that can be expressions involving distances, *x*-coordinates or *y*-coordinates. Couples of magnitudes that are in functional dependency can be exported to the symbolic window and define a function, likely to be treated with all the available tools; this can be done automatically, a functionality that was expected to help students in modeling dependencies, and that we will refer to as “automatic modeling” below.

THE EXPERIMENT

The design of a session

The classroom session analysed in this paper was the fifth of a series of sessions in the ReMath project following three sessions by which students get familiar with the symbolic window, and one in which they were introduced to the dynamic geometry window and to problems about areas. A series of tasks was conceived in which the students had to make a choice of the independent variable as a key step to get an algebraic model of a geometric dependency, in order to solve the following problem: *ABCD being a rectangle, what can be the position of a point M in order that the area of the triangle BMC is one third of the area of rectangle ABCD (Fig. 1, right)?*

The sides of rectangle were parametric ($AD=a$ and $AB=b$) in order to ensure generality and a discussion on the fact that the solution does not depend on *a*. The solution is that the points satisfying the condition belong to one of two straight lines parallel to (*BC*) crossing (*AB*) respectively in M_0 and M_1 such that $BM_0=BM_1=2AB/3=2b/3$. It is possible to reach this solution geometrically, but the way the problem was proposed to students (in coordinate geometry) and their lack of knowledge in geometry, oriented towards using a function as a model of the variable area. Five successive tasks were then proposed to the students: (1) Build the figure in the dynamic geometry window, *M* being a free point in the plane (2) Create a geometric calculation for the area of *ABCD*, and moving *M*, conjecture positions of *M* for which the area of *BMC* is one third of the area of the rectangle (3) Choose an adequate independent variable to get a model of the geometric function of the area of *BMC* (4) Use Casyopée's "automatic modeling" to get the definition of a function in the algebraic window (5) Use the algebraic window to get algebraic solutions, and then interpret these solutions in the dynamic geometry window.

Instructions were given in order that the side *AB* was parallel to the *y* axis, and the side *BC* to the *x* axis. So in task 3 the students had the choice to select for independent variable some length involving the point *M* or coordinates of *M*, but only calculations depending univocally of the *y* coordinate of *M* could be adequate variables. It was expected that students would observe that moving horizontally the point *M* does not change the area, and connect this observation with the fact that *xM* is not an adequate independent variable. After a user selects an independent and a dependent variable, Casyopée gives some feedback on whether it is possible to create a function with these data. Together with the observation of values of the variables when moving point *M*, this feedback was expected to create a milieu helping students understand the statute of variables in a function modeling a dependency.

A-priori analysis

For solving task 3, lengths involving *M* cannot be chosen as independent variables because they depend on the two coordinates. x_M can be an independent variable, but, as mentioned above, a change of x_M does not affect the value of the area of the rectangle. The version of Casyopée, still in development at that time, calculated a formula involving y_M, but after that refused to create a function. y_M is a suitable variable and the function calculated by Casyopée is $x \rightarrow a \times \frac{\sqrt{(x-b)^2}}{2}$. It was expected that the identifier *x* for the independent variable could be confusing for students. Casyopée offers other identifiers, but it was not likely that students will use this feature. In the preceding session, the independent variable was a length on the y axis, and the teacher insisted that this length could be labelled x in the function. After creating the function, the students could work in the familiar symbolic window to solve an equation. If *yM* has been chosen as a variable, the equation is $\frac{a \times |x-b|}{2} = \frac{a \times b}{3}$ but it can be different if the student choses another variable; for instance y_B-y_M is a possible choice and the equation is then $\frac{a \times |x|}{2} = \frac{a \times b}{3}$. Another difficulty was expected to emerge from the fact that Casyopée displays the solution in the non-simplified form $\frac{\sqrt{(x-b)^2}}{2}$, thus students had to interpret the two solutions in *x* as two values of y_M and to connect it to the geometric solution.

Data

The situation has been implemented in a 90-minute session in two classes. Data consisted of recordings of students' work via screen capture software, observers' field notes and students’ written assignment. Below we present briefly the work of a pair (Elina and Chloé).

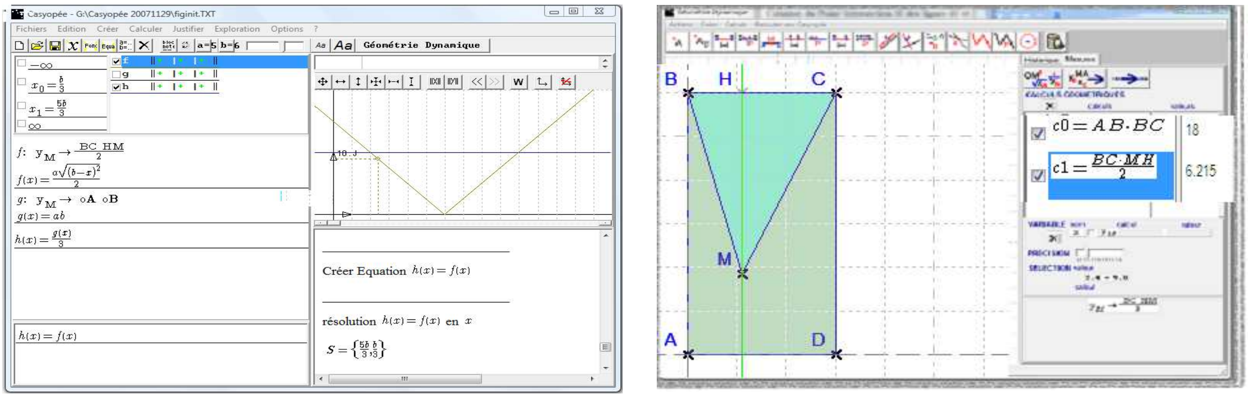

Fig. 1: Casyopée: The symbolic window and graphic tab (left) and the dynamic geometrical window and the geometric calculation tab (right).

The work of Elina and Chloé

In the first 20 minutes students built the rectangle, created a geometric calculation for the area of it and created a free point *M* initially in the plane and after some dialogue on a vertical side [*AB*]. By moving *M* on [*AB*] and evaluating numerically this area (considering the numerical values of the *a* and *b* parameters) for about 10 minutes, the students found a solution without taking numerical information of the software. They commented “This is good, this is one third of [*AB*]” and wrote their solution. After the teacher’s prompt that M is in the plane, the students explored the figure again looking for a single position of *M*, but now on the perpendicular bisector of [*BC*] without using values of the area of *MBC* calculated by Casyopée.

Then Elina proposed to create a function, but Chloé stressed that an independent variable had to be chosen first. Thus they returned to the text of the given problem and tried to identify the requested variables. Reading the message after trying the constant measure *AB*, they moved *M* to this segment. Trying *BM*, Chloé commented: “here we cannot create the variable”. After that, they tried x_B-x_M and y_B-y_M, reading the message of Casyopée (“the variable depends on *M*, it is defined over]-infinity, +infinity[“) but not creating the variable. At 50mn, the teacher told them to choose a variable and they chose x_M. Then they defined the function: $x_M \to AB \times BC$ and got a function $x \to a \times b$ in the symbolic window. After that, they defined the function $x_M \to \frac{MH \times BC}{2}$. As indicated in the a priori analysis, Casyopée calculated a formula involving y_M, but after that, refused to create a function. At 65mn, they had to recreate the figure because of a technical problem and Chloé realized that the triangle area was constant and equal to one third of the rectangle area for every position of *M* on a certain horizontal line. They commented “it is always one third… then the *y*-coordinate is what is important”. Surprisingly they again chose the variable x_M and got the same feedback as before. At 70mn, the teacher told them to test the variable y_M as indicated in the text of the problem. Casyopée indicated (*-infinity*, *+infinity*) for the domain. They were not happy and tried to find a way to redefine this domain into [0;3]. Giving up, they defined the function by way of “automatic modeling” and it was accepted by Casyopée. They tried a graphical resolution, but they were confused by the graphical window and needed to get help by the teacher. He showed them how Casyopée offers a dynamic link between a trace on the graph and the free point from which the function is built. The students observed that when they moved *M* horizontally in the geometry window, the trace does not move (Fig. 1, right).

The written report prepared by Elina and Chloé was divided in two parts: Dynamic geometry and Casyopée (i.e. the symbolic window). In the first one they describe their exploration for calculating the areas for the values a=3 and b=6 to the parameters: “in our case, the area of *BMC* must be 6. Thus we move *M* in order that the value displayed is 6. We see that there are two positions of *M* and only the *y*-coordinate has an influence on the area, the *x*-coordinate does not change the area”. As for their work in the symbolic window they write: “we chose the variable x_M” [2]. And then “we draw the functions $y_M \rightarrow AB \times BC$ and $y_M \rightarrow \frac{MH \times BC}{2}$”, copying the formula given by Casyopée and not mentioning the domain. They copy also the equation and the two solutions. They conclude: “To satisfy the condition, the *y*-coordinate of M should be y_M=5b/3 or y_M=b/3”. About the difficulties encountered, the students mention: “Finding that only *y* has an influence on the area of *BMC*, and then choosing y_M as an independent variable”.

THE ECONOMY OF LEARNING SITUATIONS: A DOUBLE ANALYSIS

A posteriori analysis from a TDS perspective

The situation was certainly productive in the sense that students could grasp the necessity of choosing an adequate independent variable in order that Casyopée will be able to express a function, but they were far from giving an algebraic signification relatively to this necessity and to the other algebraic objects involved, like for instance the parameters. Casyopée’s feedback was generally not well understood. Eventually, it conflicted with students’ views, for instance relatively to the domain. It happened that the teacher had to intervene to help going forward in the task. He tended to offer more than a technical help to students, steering them towards steps of the solution and then breaking the intended adidacticity. This was also the case in other Casyopée experiments conducted in ReMath. Thus, the influence of the provided feedback seems to be less productive than expected as regards the students’ attempts to identify key steps in their mathematical work. Relatively to the question at stake of how students could appropriate the choice of the independent variable, as well as other functionalities already encountered in the preceding sessions, and through this appropriation progress in their understanding of functions, the appreciation is then mixed: the milieu highlights actions that students can identify as steps in the solution; then students are “pushed” towards these actions; however, this does not guarantee that they acquire an appropriate understanding of these actions.

Analysis from a constructionist perspective

The issue of design was mainly materialised through the preparation of a milieu facilitating - in constructionist terms- meaning generation for function as covariation. The reported episode reveals students' diverse views of the symbolic forms provided by the tool and difficulties to relate their selection of variables to the mathematical concept of function. To analyze this divergence, we refer to *two gaps at the level of design* and try to connect it to constructionism: one has to do with the design of the environment and, in particular, *the nature of the provided feedback* and the second with *the teacher's role*. As for the first, the level of design of Casyopée at that time did not provide students with opportunities to take some actions in relation to the provided feedback. Thus, we see that students could not experiment directly with notation in Casyopée. The correct symbolic form in mathematical terms appears as a 'closed' answer pre-supposing in some way students' understanding of the standard algebraic symbolism of functional dependencies. A constructionist view on design should stress that further development of meaning generation can be facilitated if students have at their disposal a mechanism to manipulate so as to take further action based on the provided feedback (i.e. to 'do something' with the tool). Learning activity within constructionist computational media very often consists of students' engagement in debugging intentionally designed 'buggy' behaviors of objects. These objects operate as means to challenge productive meaning generation and provoke further interpretations and actions by students. Thus, constructionism should emphasize the *expressiveness* of computational environments as a design principle, i.e. design based on the use of dynamic representations that make algebraic symbols and relationships more concrete and meaningful for the students through the ability to express mathematical ideas possibly in ways that may diverge from standard mathematics (see for example the idea of *autoexpression,* which privileges the role of a programming language as a mechanism to control objects by expressing explicitly the relationships between them, Noss et al., 1997). The second point has to do with the role of the teacher. In the episode we can see that the teacher seems to be reluctant to intervene and does it only when he realizes that the students face strong problems in coping with the provided functionalities and integrating them in their activity. In a constructionist perspective, in contrast, teacher's interactions are more participatory from the teachers' side and more strategic in encouraging students to elaborate emergent ideas and generalisations.

DISCUSSION

The motivations underlying the Casyopée experiment meet at a general level those involved for the use of multirepresentational, DGS and CAS software. Its specificity is that it focuses on key points of the transition from function in a DGS to symbolic functions, aiming to facilitate students' access to symbolic forms. The task in the experiment is to find a solution of a problem. It can be explored in geometrical settings, but can be really achieved only after the transition to symbolic functions.

The milieu is then inspired by TDS, the task being challenging and the transition being conceived as a non obvious step. Feedbacks are prepared in the software in order to ensure that interaction will actually put the aimed knowledge at stake, TDS rely on an a priori analysis and uses an a posteriori analysis to compare actual procedures of students to the a priori expected procedures to bring evidence that the milieu is adequate for the targeted knowledge. That is what it was aimed in the Casyopée experiment. However, as indicated above the appreciation is mixed: the interaction seems to produce effects in terms of action in the environment, but not to really make sense for the students. The constructionist interpretation points to an important fact: Casyopée is a mathematical tool, and most of the feedback it provides supposes algebraic knowledge, or coordination between geometry and algebra, that is precisely at stake. In this vein, constructionist analysis brings to the fore issues of tool design emphasising the importance of design choices allowing students' meaningful use of the available infrastructure by forging connections between students' action and tool formalism. The interventions of the teacher constitute a point of common interest in the double analysis. They are seen by constructionism as participatory and strategic in enhancing students' exploratory activity. In contrast, TDS cares for adidacticity that could be broken in these interventions, by way of "Topaze" effects. However, in the Casyopée experiment, it happens that total adidacticity would have led the students to an impasse.

Constructionist and TDS analyses of learning situations in the Casyopée experiment in part converge when they consider a milieu and in part diverge because they have different conceptions of this milieu. TDS analysis is oriented towards evaluating the reproducibility of situations of learning aiming a given knowledge, and constructionist analysis towards identifying occurrences of progression of meaning. However, this "double analysis" is clearly deeper and helps to look at the economy of learning situations about functions with computers as a particularly complex question. On the one hand, the multiplicity of interconnected representations of functions, of students' possible actions on these, as well as of students' understanding of these representations and actions is an obstacle to the possibility of a controlled milieu, and of adidacticity consistent with TDS. On the other hand, relying exclusively on uncontrolled meaning generation would question the extent to which connections can be made between knowledge built by interacting with the milieu ("knowing") and the standard mathematical knowledge at stake ("knowledge").

NOTES

1. "Representing Mathematics with Digital Media", 6th FP, IST-4-26751-STP, 2005-2009 (http://remath.cti.gr)

2. Actually, *xM* was the label of the button allowing choosing a variable, which explains why the students mention this label, while being aware of *yM* being the right choice. This label changed in subsequent versions of Casyopée. The

design decision at the time was to implement key actions at Casyopée's interface by way of buttons like in DGS. The difficulty was to find icons that could accurately represent the nature of the action.

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