DEVELOPING AN INTUITIVE CONCEPT OF LIMIT WHEN APPROACHING THE DERIVATIVE FUNCTION

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The central idea of calculus is the concept of limit. German secondary school curricula claim to introduce the concept of limit in an intuitive way refraining from a rigorous mathematical definition. However, it is unclear what can be regarded as a "good intuitive basis". The scope of approaches chosen by teachers varies from talking about only a few concrete examples to a mathematically rigorous approach as done in university lectures. After the discussion of typical student problems we present a DGS-based activity to support the introduction of the derivative function fostering a dynamic concept of limit. Aside from the step from derivation as a local phenomenon to the global view of the derivative function we also show how the activity can be used to visualize and talk about the variety of limit processes.

Keywords: conceptualization processes, DGS, interactive learning activity, concept of limit, teaching and learning of calculus

INTRODUCTION AND RATIONALE

The problems of contemporary calculus courses at secondary school mainly result from the tension between learning or teaching of routines and the development of a structural understanding of the underlying concepts. E.g. Tall (1996), p. 306, writes:

> If the fundamental concepts of calculus (such as the limit concept underpinning differentiation and integration) prove difficult to master, one solution is to focus on the symbolic routines of differentiation and integration. […] The problem is that such routines very soon become just that — routine, so that students begin to find it difficult to answer questions that are conceptually challenging.

In fact this problem exists since the introduction of calculus to German school curricula: For example Toeplitz (1928) urges to remove calculus from school if teachers are not able to bring out more than the teaching of mere routines.

The fundamental concepts of calculus (e.g. the concept of limits, derivation and integration) are mathematically advanced. Therefore teachers are always required to make didactical decisions about what to teach in a visual-intuitive way and what to teach in a mathematically rigorous way. Of course, this is not a bipolar problem; instead there are all kinds of intermediate stages. Especially in the context of calculus concepts intuition is often misleading and remaining mathematically consistent is not trivial on an intuitive basis. Otherwise, mathematically rigorous approaches do not automatically lead to a deeper understanding.

For example the secondary school curriculum in Berlin, Germany, explicitly demands an intuitive or propaedeutic approach to calculus concepts at the end of grade 10. This is established through introducing a special „Modul“ called „Describing change with functions“ (Jugend und Sport Senatsverwaltung für Bildung 2006a). In this context Hoffkamp designed and investigated several DGS-based activities to support such a qualitative approach (Hoffkamp 2009, 2011), which could lead to a sustainable intuitive basis of certain calculus concepts.

Considering the central role of the concept of limit within calculus, the described situation needs particular attention. On the one hand the Berlin secondary school curriculum for grade 11 mentions that the concept of limit can only be taught in a propaedeutic way since the necessary exact notions (series, convergence tests) are not available to the students. On the other hand, the teachers are required to introduce the derivative as limit of the difference quotient (Jugend und Sport Senatsverwaltung für Bildung 2006b).

From a mathematical point of view one could think that doing calculus without the exact notion of limit is impossible. Of course this is not true, since it is quite natural to approach mathematical concepts or problems in an intuitive way. Only in the process of doing mathematics ideas and notions are determined more and more precisely. From a learning theoretical or psychological perspective this is reflected by the so-called „orthogenetic principle“ (Werner 1957, p. 126):

> Developmental psychology postulates one regulative principle of development: it is an orthogenetic principle which states that wherever development occurs it proceeds from a state of relative globality and lack of differentiation to a state of increasing differentiation, articulation and hierarchical integration.

Sfard (1991) describes the process of mathematical concept formation. Starting with operational conceptions (e.g. functions as computational processes, rational numbers as results of division of integers) structural conceptions (e.g. function as set of ordered pairs, rational numbers as pairs of integers) develop, the latter leading to the establishment of abstract objects.

Fischbein (1989) offers another perspective on the orthogenetic principle as well as the process of mathematical concept formation. He describes how, during the process of mathematical abstraction, mental models of the abstract concepts develop in the learners mind. According to him these intuitive models, tacitly or not, influence the way we conduct mathematical reasoning processes. These tacit models are one reason for the difficulties students are facing in the process of learning mathematics. He suggests allowing the students to consciously analyze the influence of those tacit models and in this way to allow them to avoid the development of misconceptions.

Taking the above into account means to realize the necessity of a profound intuitive basis for the limit concept that could lead to the development of an object view (in the sense of Sfard, 1991) of limits and derivative. DGS appears to be a good means in this context. In this article we present an activity using DGS-based interactive visualizations fostering a dynamic idea of limits and leading to an object view of derivation. In this context Sfard (1991, pp. 6-7) mentions that

> Visualization [...] makes abstract ideas more tangible, and encourages treating them almost as if they were material entities. [...] Visual representation is holistic in its nature and various aspects of the mathematical construct may be extracted from it by „random access“.

For example Hoffkamp (2011) observed positive effects for the development of the function concept and the object view of functions resp. functional thinking by using interactive visualizations in the context of propaedeutics of calculus.

In the following we will describe the conceptual change approach and the potential of DGS-based activities. We will give an example of an activity that is related to our rationale. After that we will present the research questions that guide our recent and future work.

Spontaneous conceptions of limit and conceptual change

The above considerations conform to a genetic view of learning as described by Wagenschein (1992). Especially for the process of conceptualization, a theoretical perspective like the conceptual change approach is helpful to design activities like the one described in this article, and to understand the students' learning processes. The conceptual change approach itself is a genetic learning theory. A description can be found in Verschaffel & Vosniadou (2004). Conceptual change does not mean to switch from one concept to another by replacing the old concept by a better new one. However, conceptual change is the process of reintegration and reorganization of cognitive structures in order to develop mental conceptions and to activate the appropriate conceptions dependent on given contexts:

> More specifically, a number of researchers have pointed out that even in the case of the natural sciences conceptual change should not be seen in terms of the replacement of students' naive physics with the „correct“ scientific theory but in terms of enabling students to develop multiple perspectives and/or more abstract explanatory frameworks with greater generality and power. (Verschaffel & Vosniadou 2004, p. 448)

For the process of conceptualization it is important to know about the students' spontaneous conceptions and to build on them. In fact the students' spontaneous conceptions can be considered a learning opportunity and a starting point for further development by dealing with them explicitly (Prediger 2004).

Therefore we tried to find out about the students' spontaneous conceptions of limits. We asked students at the end of grade 10 and 11 to write a letter to an imaginary „clueless“ friend explaining the mathematical notion of limit. We present two excerpts from letters here. One student wrote:

> The limit is the outermost value of a number range. For example if one says „all numbers from one to five“ then one and five are limits.

Another student wrote:

> Considering the graph of a function over a large interval one can observe that some functions come closer and closer to a certain value. The function tends only to this value, without drifting away again. However, the function does not reach or exceed it.

The first excerpt shows that the student considers limits as bounds of intervals and formulates a static conception of limit. The second excerpt reflects the student’s experiences with limits in connection with asymptotic behaviour. Although the student formulates a dynamic conception of limit, his conception is not elaborated since limit processes seem to be always „monotonous“ and limits „cannot be reached or exceeded“.

These observations are confirmed by the work of Cornu (1991) who described typical spontaneous conceptions of limits. In fact all limiting processes like the concepts of continuity, differentiation or integration contain similar cognitive problems: To overcome or integrate the spontaneous conceptions in the learner’s individual concept.

THE USE OF DGS-BASED ACTIVITIES

As already mentioned we think that DGS is a good means to establish an intuitive basis of the concept of limit. With respect to the mentioned spontaneous conceptions and the conceptual change approach, DGS-based activities could not only help to develop a dynamic view of limit and limit processes, but also help to develop a more elaborated conception of limit by visualizing the variety of limit processes. Therefore we combine interactive visualizations with special tasks stimulating verbalization and exploration processes. Especially the role of verbalization as mediator between the representations and the students’ mental concepts when working with interactive visualizations was pointed out in the work of Hoffkamp (2011) and based on Janvier’s work (1978).

Which role of the computer do we focus on? At first, we benefit from the various possibilities of visualizing mathematical concepts. Therefore, we make use of the possibility to visualize a holistic representation in contrast to a linear order of mathematical content (see also Sfard 1991). Moreover we add interaction to enable learners to explore the interactive activities without negative consequences. Schulmeister (2001) states that especially the lack of negative consequences and the possibility to work self-determined have a positive effect on the learner’s motivation.

While giving the opportunity for exploration we use the computer to „restrict the actions of learners and thus help them to develop appropriate mental models of representation“ (Kortenkamp 2007, p. 148). In this sense visualizations play a heuristic role and can be used before exact mathematical notions or concepts are available.

THE ACTIVITY “TOWARDS THE DERIVATIVE FUNCTION”

In the following we present a DGS-based learning activity. It introduces the derivative function as an object and its relation to the original function. This activity is meant to be exemplary. It shows how conceptualization processes in the context of the concept of limit can be supported by using special DGS-based activities. Our idea of using the difference quotient and its extension by continuity for an object-based approach to obtaining the derivative function has already been mentioned by Mueller and Forster (2003) quoting Yerushalmy and Schwartz (1999). However, they did not explicate the full didactic potential of this approach and did not use the dynamic approach towards limit that can be fostered by a DGS-based activity.

The two focal points for our activity are emphasizing the difference quotient and permitting a look on various limit processes that constitute the analytical step. The difference quotient is not only a (theory generating) precursor for the differential quotient and the derivative (the way it is often used in schools). It is the concept that has a direct relation to reality through the concept of average change (of speed etc.). Therefore it is evident for students and is thus worth taking a closer look at.

The activity is meant to be used when the derivative of a function at a certain value is already known to the students. However, the students have no elaborate limit concept so far. This is usually the case at the start of grade 11 in secondary schools. The derivative function, however, is not known to the students yet. The way from a local (derivative at a point) view on derivation towards a more global, object based view (derivative function) shall be supported, while several limit processes get examined on the way. This is in line with our idea of broadening the view on limits as well as bringing about an object view on the derivative function.

We plan to introduce the activity in a grade 11 course at a secondary school in Berlin at the end of August 2012. Videography of pairs of students will be done and analyzed. Results shall be presented on the CERME 2013.

The activity can be found at

http://www2.mathematik.hu-berlin.de/~hoffkamp/Material/ableitungsfunktion.html.

Except for JAVA and a common web browser no special software is needed to use the activity. Therefore the technical overhead is pretty low. The basic structure of the activity is a HTML website with an embedded interactive JAVA applet and text based instructions and tasks.

Description of the activity and didactical analysis

The activity consists of three separate worksheets or tasks. In this section each task will be described and didactically analyzed in more detail. Figure 1 gives a general idea of what one of the tasks looks like. The text above the applet gives an introduction to the task. The text next to the applet poses special questions and tasks that also ask students to verbalize their thoughts and observations and are meant to help the students go through the conceptual change process we intend to initiate. This also relates to Fischbein. Verbalization makes tacit models and intuitive concepts accessible to a conscious process of reflection.

The tasks are consecutive. Different predetermined functions f can be chosen in every task to make sure there is not just one but many graphs available. The restriction on predetermined functions allows a broadening of the view while at the same time focusing on certain sustainable examples (see also Kortenkamp 2007). The functions were chosen as examples of certain classes of functions - symmetric and non-symmetric, polynomial and trigonometric functions and in tasks 2 and 3 also the (at the origin) non-differentiable absolute value function.

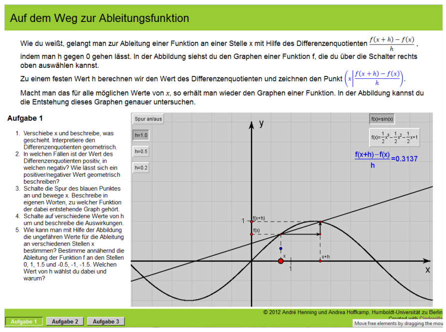

Figure 1: General overview of a worksheet.

The first task takes up the so far local ideas of the derivative at a point and differential quotient as the limit of the difference quotient. In the second task a first object view on the derivative function is reached. Also the limit process observed is changed as will be seen later. The third task introduces yet another variation of the limit process while the object view on the derivative function remains in focus.

The first Applet (see figure 2) offers a process view on the functions we will take an object view on in tasks 2 and 3. For three fixed values of h the term $\frac{f(x+h)-f(x)}{h}$ (the difference quotient) is evaluated at a certain x-coordinate. The x-coordinate can be changed by dragging the big red point on the x-axis. The point $\left(x\middle|\frac{f(x+h)-f(x)}{h}\right)$ is always printed in blue. The difference quotient is already known to the students. So far they only evaluated it for a single fixed value of x and varying values of h to gain the derivative at a point. If tracing is activated one gets a trace of the resulting points which forms the graph of the function g with $g(x)=\frac{f(x+h)-f(x)}{h}$ (for fixed h). This way we have a point wise (process) view of the construction of the graph. We change between processes by varying h. The resulting function becomes a better approximation of the derivative function for smaller values of h. It can be seen how the new object develops as the result of a process of point wise evaluation of the difference quotient. The new object will allow the students to develop a more global view. The given tasks shall support this. The task “What is the meaning of the value of the difference quotient?” is particularly interesting. The short use of the activity by students already showed, that there seems to be an epistemological obstacle in the following sense: Students reason a positive slope of f on an interval from a positive value of the difference quotient. This reasoning however is only possible through the analytical step; the core achievement of school analysis. The value of the difference quotient just tells us that there must be a point in the regarded interval, where f achieves this value. These observations lead directly to the mean value theorem.

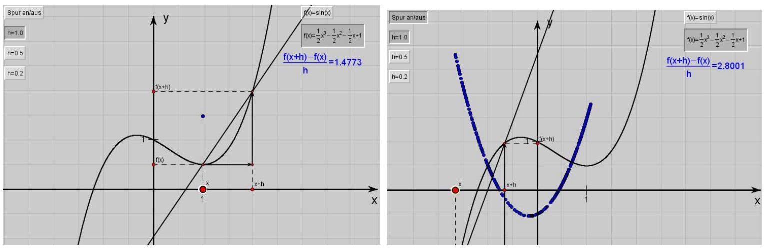

Figure 2: Applet 1 without and with tracing within the first task.

The second and third applets are pretty similar in their construction, which is why only one of the two is pictured here in figure 3. They both show graphs of functions that can be manipulated. In applet 2 there is only one function namely g with $g(x)=\frac{f(x+h)-f(x)}{h}$. In addition to that applet 3 also visualizes the function k with $k(x)=\frac{f(x+h)-f(x-h)}{2h}$.

Switching over to applet 2 we have a new situation. The function that developed as the result of a process (of changing the x value) in applet 1 now exists as a single entity. We evaluate the difference quotient for all values of x simultaneously now (note that this is not the difference quotient function as that would be $g_x(h)=\frac{f(x+h)-f(x)}{h}$ for a fixed value of x). At the same time we no longer have fixed values of h but can change those using a slider. If the value of h is changed, the graph of g moves as a whole. h cannot be chosen as zero in our applet (as g is obviously not defined for h=0). However, it is possible to choose h very close to zero - the resulting graph is a very good approximation of the graph of the derivative function for differentiable f. The students observe a family of curves that converges to a limit function. According to Sfard (1991) working with families of curves or equations with parameters is already a step towards reification. Limit becomes something more dynamic in this context as the students observe and describe the limit process that leads to the derivative function for differentiable f. Two tasks are central here. First, the students are asked to evaluate the properties of the approximated derivative function (zeros, monotony etc.) and to infer to properties of the original function, thus emphasizing the connection between a function and its derivative. Second, the students discover that the absolute value function is not differentiable for x=0 and are asked to explain why, supported by a dynamic visualization of a limit process.

The third applet is aimed at the development of a more elaborated conception of limit. In contrast to task 2 we now take a look at not only one but two different limit processes. We show that different limit processes can lead to the same limit. The idea is to prevent or change a very restricted view on limits as could be seen in the student letters described above. An object view on the functions involved is necessary now.

One observed object is the function g, already known from task 2, the other is the above mentioned function k with $k(x)=\frac{f(x+h)-f(x-h)}{2h}$. k represents a symmetric differential quotient. It has some interesting characteristics. For example, if f is a symmetric function, k has a local extremum where f' has a local extremum for any given value of h. For g this is obviously not the case. The students are asked to describe and compare the functions g and k and to explain their observation geometrically (by using secants) and algebraically (by comparing both forms of difference quotients). One observation is that k converges faster to f' than g. Therefore the symmetric difference quotient is better for numerical computations. For polynomial functions this can even be proved with students. The students may discover that convergence is not always the same and that different processes may converge at different speeds. The observed properties of the usual difference quotient can be transferred to and verified for the symmetric difference quotient. It is hoped that these examinations encourage an object view of function as well as a process view of limit and lead to a higher level of integration of the more advanced concepts.

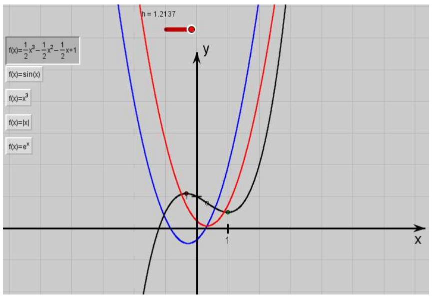

Figure 3: Applet 3 within the third task.

RESEARCH QUESTIONS AND CONCLUSIONS

Our future work will focus on the following research questions:

- In what way can a dynamic-visual approach as depicted in this paper support the development of a sustainable conception of limit and related mathematical concepts?
- What conceptions of limit do students develop if confronted with activities as the one presented?

- Which pre-conceptual terms do students use when talking about limit processes and related mathematical concepts?
- What epistemological obstacles can be uncovered through analysis of students' work and how can they be used as learning opportunities?

These questions lead to an important future task: development and analysis of further activities – technology based or not.

As mentioned above, the activity will be used at a school at the end of August 2012 and videography of students will be done. This activity is meant to be exemplary and in no way meant to offer a whole concept for the introduction of the limit concept. More work in the field of ICT-based concepts and activities, also in the way of mathematics education as a design science (Wittmann 1995), is necessary for the elaboration of a full concept for the introduction of limit.

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