TEACHING INVERSE FUNCTIONS AT TERTIARY LEVEL

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This study is a part of an ongoing research that attempts to explain the relationship between the teachers' instructional practise and students' learning in the context of functions and function inverses. The question in this paper is how the use of technology as a pedagogical tool may contribute to the understanding of the inverse function concept. An engineering student group (n =17) was taught functions and inverse functions with the assistance of GeoGebra. In our theoretical framework we apply Variation theory together with the theory of Concept image and Concept definition. The data were gathered by doing a pre and post test concerning inverse functions. Our experiment revealed that students' concept images in the post test were more developed compared with the results in the pre test.

Key words: Concept definition, Concept image, Inverse functions, Technology, Variation theory

INTRODUCTION

The function concept is a central but difficult topic in mathematics and therefore it has received considerable attention in mathematics education (Akkus, Hand & Seymour, 2008; Ponce, 2007). Strong understanding of the concept of function is crucial for any student hoping to understand calculus, which is a critical course for the prospective teachers, engineers, and mathematicians.

Functions have different faces, and to make students aware of that is a pedagogical challenge for teachers in mathematics. A number of studies have been conducted concerning students' understanding of the functions at the tertiary level, confirming a frequent inconsistency in students' conceptions of function and the definition of function (e.g., Thomas, 2003; Thompson,1994; Vinner & Dreyfus, 1989). Vinner and Dreyfus (1989) conducted one study, showing that tertiary students during a course in calculus, even when the students were able to correctly formulate the definition of function, could not apply the definition of function successfully.

Even (1992) conducted an investigation of prospective secondary math teachers' understanding of inverse functions. She found that many students conceptualized a function inverse using the notion of 'undoing' (p. 557). "'Undoing' is an informal meaning of inverse function which captures the essence of the definition" (Even, 1992, p. 557; Wilson et al., 2011).

Bayazit and Gray (2004) investigated student learning of function inverses from two teachers, Ahmet and Mehmet. Ahmet focused his instruction on the idea of inverse "undoing" operations, whereas Mehmet on algorithmic and procedural skills (Bayazit & Gray, 2004). Students were given pre test and post test to evaluate their understanding about inverse functions before and after the classroom instruction.

Results from the post test indicated that more students from the class of Ahmet were able to answer a question regarding the domain and range of inverse functions correctly using verbal explanation. In Ahmet's class 25% of the students chose to take a global approach in reflecting the function across the line $y = x$, while no students in Mehmet's class used this method. The authors conclude that in order to grasp the concept of inverse function, students would be given the opportunities to experience conceptually focused tasks (Bayazit & Gray, 2004, p. 109).

Can technology as a pedagogical tool help students to understand different faces of the concept of the inverse function? Technology is becoming increasingly used at teaching of university mathematics but there are still few studies which have examined technology-assisted teaching at the university level, even though university mathematics teaching has been changing quickly during the past two decades (Attorps et al., 2011; Lavicza, 2006; 2007; Zimmermann, 1991).

Theoretical framework

In our study we apply two theoretical frameworks, the Variation theory and the theory of Concept image and Concept definition. We consider these two theoretical frameworks to be complementary. The theoretical constructs of concept image and concept definition have proven to be a useful analytical tool for nearly three decades (Tall & Vinner, 1981). According to Vinner (1991) and Tall (1999), each mathematical concept is associated with concept definition and concept image. The concept definition can be the stipulated as a definition assigned to a given concept. The concept image, on the other hand, is a nonverbal representation of any individuals understanding of a concept. It includes the "visual representations, the mental pictures, the impressions and the experiences associated with the concept name" (Vinner, 1991, p. 68). We agree with Vinner in believing that many mathematics instructors generally would imagine that their students' concept image is growing out of a delivered concept definition in class and normally supplied with a textbook definition. In our study, it is our strong belief that by using the GeoGebra software, we have affected the student's concept image of functions and function inverses.

Another interesting framework in our study is Variation theory. Teaching and learning research has found that ways of experiencing something are essential to what learning takes place (Shulman, 1986). Marton & Booth (1997) stated that qualitatively changed ways of experiencing something is the most advanced form of learning. If we can describe learning as coming to experience something in a changed way, we should also acknowledge that experiencing something must require the ability to discern this new way of seeing the experience. Central in Variation theory is an assumption that variation is needed to discern aspects of object of learning not previously distinguished by learners. According to this theory the most powerful factor concerning students' learning is how the object of learning is handled in a teaching situation. Marton et al., (2004, 16) have identified four patterns of variation in a learning object: contrast, generalization, separation and fusion. They are described as follows:

Contrast: … in order to experience something, a person must experience something else to compare it with.

Generalization: … in order to fully understand what ‘‘three’’ is, we must also experience varying appearances of ‘‘three’’,…’’

Separation: In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant.

Fusion: If there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously.

According to Leung (2003), these patterns of variation create opportunities for the students to understand the underlying formal abstract concept. In order to generate the patterns of variation, we use the dynamical nature of the GeoGebra software, which has the “ability to visually make explicit the implicit dynamism of ‘thinking about’ mathematical, in particular geometrical, concepts.” (Leung, 2003).

The purpose of the study

The aim of this study is to investigate if the technology-assisted teaching of functions and function inverses at the university level can contribute to the development of engineering students’ understanding of the concept of function and function inverse. The study investigates the following research questions:

1) How can the patterns of variation be visualized by using GeoGebra when teaching the concept of function and function inverse?
2) Which qualitative differences between the students’ concept image of the concepts function and function inverse could be distinguished in pre and post test results?

METHOD AND DESIGN OF THE STUDY

The study took place during one teaching session in mathematics at a Swedish university. A total of 17 students were involved. They were all students at the engineering program, studying the course Calculus in one variable. The data were gathered by analysing the teaching sequences during the lecture and by doing a pre and post test. In the analysis of the test results we started by making an easy quantitative overview. Then we continued with a qualitative analysis of the outcomes of the students’ answers.

The pre and post test

The test contained five questions, including both conceptual and procedural ones. Students had maximum 30 minutes to do the test. It was not allowed to use any technical facilities. In this paper we focus on the following three questions from the test:

1. How would you explain if someone asks you: What do you mean by the concept of function? You may like to explain by drawing a picture.
2. How would you explain if someone asks you: What do you mean by the concept of inverse function? You may like to explain by drawing a picture.
3. How would you explain if someone asks you: What do you think are necessary conditions for a function to have an inverse? You may like to explain by drawing a picture.

In order to categorize the answers to the questions above based on their quality, we coded the answers as following: no explanation, incorrect explanation, acceptable explanation, good explanation and excellent explanation.

RESULTS

We used the pre test results as a starting point to design our lecture. In order to create different teaching sequences that could encourage students to discern varying aspects of the object of learning, we applied the ideas of the variation theory in the context of the free dynamic mathematics software GeoGebra.

Teaching sequences

Teaching sequences were implemented in an ordinary lecture with a teacher manipulating the computer and students observing the screen. In the first application of GeoGebra (Figure 1), we visualize, by using dynamically the vertical test, that the graph is a function (the most left picture in Figure 1). By applying the horizontal test, also dynamically, on this function we can see that it doesn't possess an inverse (the picture in the middle in Figure 1). However, by shrinking and moving the domain of the function we can find an interval where the function is invertible.

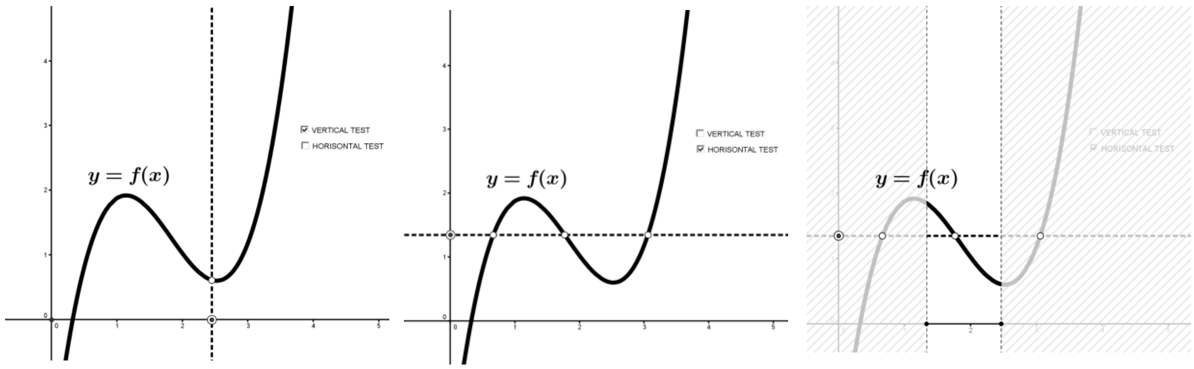


Figure 1: The domain of the function whose graph passes the vertical test has to be adjusted in order to make the graph to pass the horizontal test.

In order to experience the pattern of variation, *generalization*, i.e. to experience that the vertical test works in all positions for the given function, in Figure 1 (the left picture) we moved the vertical line through several points in the domain of the function. In the teaching sequences related to Figure 1 (the middle picture), the students were given opportunities to experience a *contrast*, i.e., to discern that the horizontal test both works and for some points doesn't work on the same graph depending on the position of the horizontal test line. In the right picture in Figure 1, we illustrated the pattern of variation called *separation* by changing both the length and the position of the interval representing the domain of the function. In this way the students were given the opportunity to experience one of the necessary conditions for a function to have an inverse, namely, being strictly monotonic.

The second example (Figure 2) should help the students to understand how to plot a given invertible function and its inverse in the same coordinate system. We use here the dynamical nature of GeoGebra to show how an arbitrary point on the function graph is reflected in the line y = x.

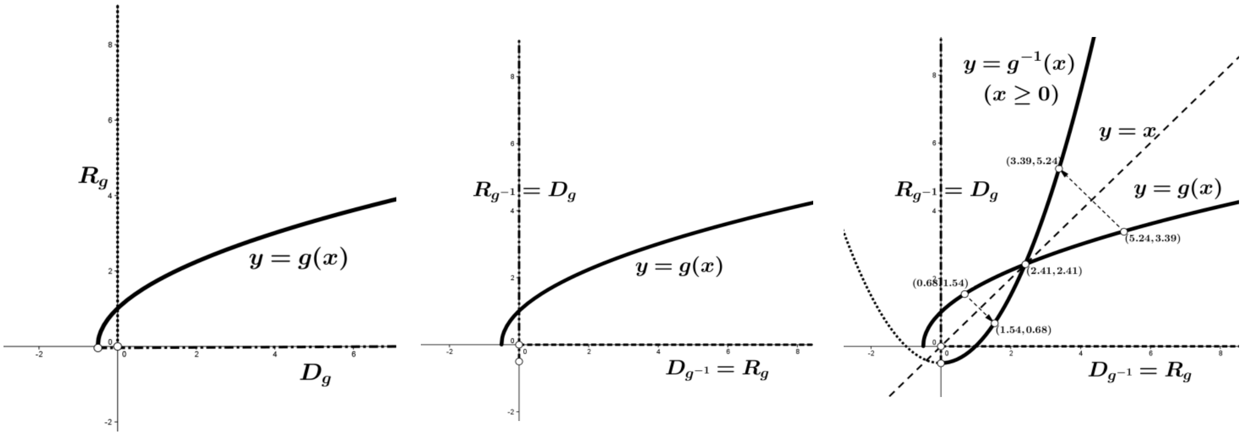


Figure 2: Plotting the graph of function g(x) and its inverse $g^{-1}(x)$.

The teaching sequence illustrated by Figure 2 should help the students to understand how to plot a given invertible function and its inverse in the same coordinate system. They were given the opportunity to experience the pattern of variation – *fusion*. The critical aspects that they could discern simultaneously were the following three: reflecting of the function graph through the line y = x, in the corresponding points x- and y-coordinates switching the positions and observing that the domain of the inverse must be restricted.

In the third presentation (Figure 3) we wanted to illustrate an informal conception of inverse function, "undoing", which captures the essence of the definition.

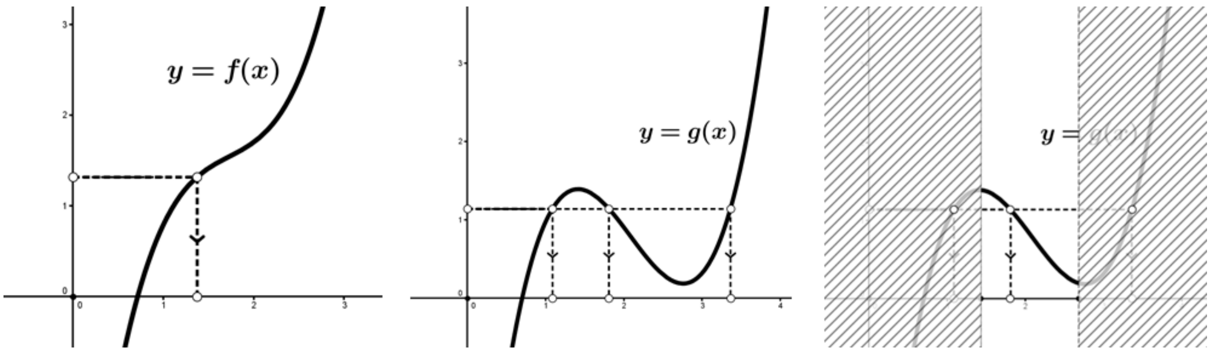


Figure 3: Illustrating of the "undoing" process.

In Figure 3 we wanted to illustrate the concept of "undoing". In the first two pictures to the left we created the opportunity for students to experience the pattern of variation – *contrast*. In the third picture to the right we separated the crucial condition concerning the existence of the inverse, namely, strictly monotonicity of a function connected to the informal notion "undoing".

Qualitative differences distinguished in pre and post test results

The Geogebra lecture began with a brief review of the concept of function, familiar to students from previous mathematics courses.

Figure 4 shows an overview of how students responded to the questions in the pre and post tests according to our criteria described in the Methods section.

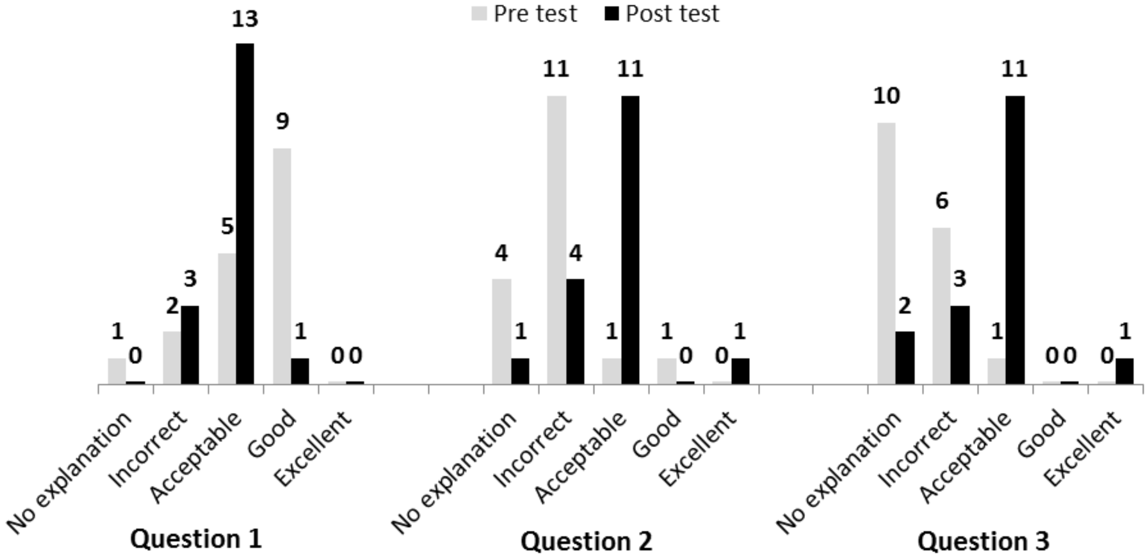


Figure 4: Students' pre and post test responses to the test questions.

In order to find qualitative differences between pre and post test results we selected sex students' responses to analyse the changes in their concept image more deeply.

The first question in our test focused on the perceptions students had about the function concept. It turned out that we could not notice any direct qualitative differences between pre and post test results. The students already had adequate pre knowledge of this notion.

The second question in our test centered on the students' conceptions of inverse functions. When we analysed the students' responses in the pre and post tests we noticed qualitative differences in their concept image. Some of the students responded in the pre test as follows:

Student 1: The inverse function is a function that is undoing another function.

Student 2: It is a reflection of a function

Student 3: It is a mirror function

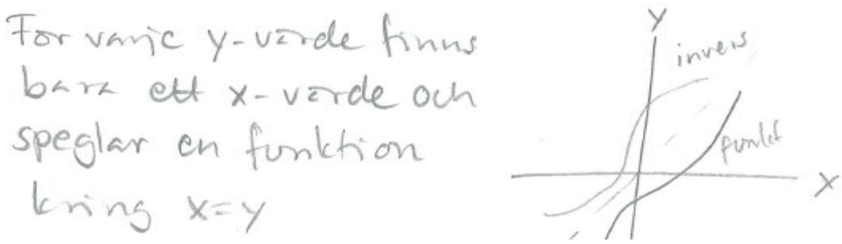
Student 4:

In the post test the same students' conception was:

Student 1: The inverse function is a function that is undoing another function or taking back an original function. If I take the starting value x, expose it to a function to form a final product y, so I can take this final product y, expose it to the inverse of the first function and get x that was my start value.

Student 2: An inverse function reflects the function in the line x = y

Student 3:



Student 3 says: For each value y is only one value x and reflects a function around x = y

Student 4:

Student 4 says: It is the reflection in itself of the function

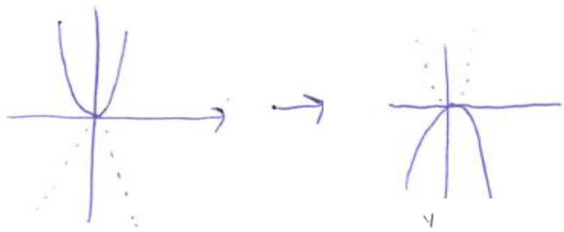
The students' pre and post test responses to the second question reveal that they (S2, S3, S4) often have an intuitive conception about inverse functions as some kind of mirroring. However they often lack the full comprehension of why and where the mirroring should be performed. The results above also show that one of the students (S1) became able to completely explain the notion "undoing".

The third question in our test focused on conditions that must be satisfied for a function to have an inverse. Some of the students responded in the pre test as follows:

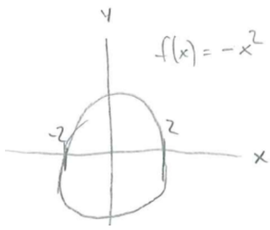
Student 1: No explanations.

Student 2: What is required is that the function is really a function and it must be symmetric.

Student 5: That you should come back to the original position and that it is "vice versa".

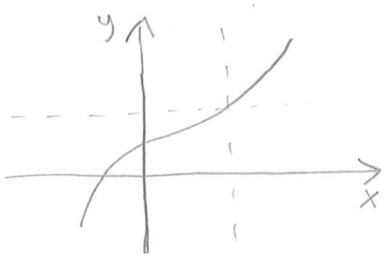


Student 6: A function can only have one value on y and x axis because otherwise you cannot come back to the same place again.



In the post test the same students responded this way:

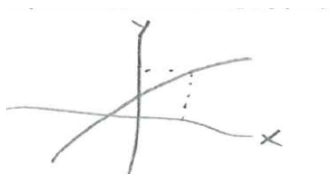
Student 1: One value x must correspond to one value y and one value y should correspond to one value x. If one draws a function in a coordinate system the graph of the function is neither to cut an imagined horizontal line more than once, nor to cut an imagined vertical line more than once.



Student 2: Each value x should only have one value y, and each value y will only have one value x.

Student 5: One horizontal line and one vertical line will only be allowed to have one intersection in the graph.

Student 6: There should be only one value x and one value y on the graph. In the figure there is both an inverse and a function.



Unlike the poor pre test responses, the students' answers in post test show that most of the students after the lecture could give a rather good explanation about the conditions that are necessary for a function to have an inverse.

DISCUSSION

Patterns of variation i.e. generalization, contrast, separation and fusion create learning space for the students to understand the underlying formal abstract concept. According to Leung (2003), when engaging in mathematical activities or reasoning, one often tries to comprehend abstract concepts by some kind of mental visualization of conceptual objects in hope to discern patterns of variation.

By continuously moving the vertical line through several points we could present the general idea of function in terms of "vertical line test" (generalization). In the similar way we created opportunities for students to experience a contrast by using the standard horizontal line test on a given function. We moved the horizontal line continuously so that students could experience the contrast between functions having an inverse and functions not having an inverse by simply counting the number of intersection points (one or more). We illustrated separation by changing the length and the position of the interval representing the domain of the function, one at a time. In this way the students were given the opportunity to experience one of the necessary conditions for a function to have an inverse, namely being strictly monotonic. In order to get the students to experience fusion several critical aspects could be illustrated simultaneously for instance by reflecting the function graph through the line y = x.

Analyzing the pre and post test results we could notice that the students' concept image of the inverse function had developed. For example, they were able after the lecture to explain why and where the mirroring should be performed. Furthermore, they were able to completely explain the meaning of "undoing". We could also notice that most of the students after the lecture could give a satisfactory explanation about the conditions that are necessary for a function to have an inverse.

As already mentioned, this study is a part of an ongoing research that attempts to explain the relationship between the teachers' instructional practise and students' learning in the context of functions and function inverses. Our ambition in the future research is to further explore already collected data which also involve the pre and post test results for a control group, as well as qualitative results from the final exam.

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