REVISITING UNIVERSITY MATHMETICS TEACHING: A TALE OF TWO INSTRUCTORS

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This paper examines two lessons in an infinitesimal calculus course given by two instructors with different agendas. The lessons were based on the same lesson plan but decisions the instructors made prior to class took the lessons in substantially different directions. Using Schoenfeld's framework for analysing decision-making we describe the resources, beliefs, attitudes and teaching goals which led and guided the instructors. These findings provide an insight into the different agendas and considerations underlying the instructors' decisions and the subsequent course taken by their lessons. On the basis of this analysis we will reflect on the impact that background, teaching experience and orientations have on mathematics lessons at the university.

Keywords: Undergraduate mathematics teaching, Real analysis, Teaching approaches, Reasoning and proofs, Mathematical language

INTORODUCTION

University math lectures are often accompanied and complemented by lessons given by teacher assistants (TAs). In these lessons the TAs usually review material that was taught in the lectures, present examples and solve exercises. This is also the case in an infinitesimal calculus course for first year math students at a leading university in Israel. Every week students of this course divide into smaller groups and attend a TA lesson of their choice. These lessons are supposed to a follow a lesson plan written according to the lecturers' needs and students are told they can attend any TA lesson because all the TAs teach more or less the same lesson. In this study we examine and compare two of these lessons given by two different TAs and based on the same lesson plan. These two lessons showed substantial differences, which were evidently the result of decisions which were made by the two TAs prior to the lesson and took the two lessons in very different directions. Thus the question this paper reflects upon is: *What led each of the TAs to interpret the lesson plan in the way that he did.*

In order to investigate this question we will employ Schoenfeld's resources, orientation and goals (ROG) theory on in-the-moment decision-making processes, and extend it to the decisions made prior to the lessons. By inferring from the TAs' actions in class and by analysing the TAs reflection on their own preparation for the class we will describe the resources, orientation and goals underlying the TAs' decisions and consequently the way the lessons evolved. On the basis of this analysis we will reflect on the impact of the TAs' background, agendas and teaching experiences on their lessons. An ongoing follow-up study incorporates the important aspect of students’ learning, which will not be discussed here.

LITERATURE REVIEW

In recent years there has been a surge in research on university mathematics education and on the practice of university mathematics instructors. Studies have criticized university teaching and also set is some challenges (e.g. Alsina, 2002), identified different lecturing styles (e.g. Weber, 2004) and described levels of pedagogical awareness (Nardi, Jaworski, & Hegedus, 2005). Some studies on the practice of university lecturers have employed Schoenfeld's ROG theory to analyse decision making during lectures (Hannah, Stewart, & Thomas, 2011) and internal conflicts of mathematics lecturers (Paterson, Thomas, & Taylor, 2011). However, according to various members of the research community, there still is a shortage of empirical research describing and analysing teaching practice, what teachers do and think daily, in class and out, as they perform their teaching work (Speer, Smith, & Horvath, 2010) whereas most research on this topic consists of researchers' suggestions for how the pedagogy in the advanced mathematics courses could be improved while there have been relatively few studies on how advanced mathematical courses are actually taught (Weber, 2004).

In contrast there are much more studies in primary and secondary math education about how math is been taught. In particular, the gaps between intended and implemented curriculum has been studied (e.g. Even & Kvatinsky, 2010).

THEORETICAL FRAMEWORK

To examine and analyse the decisions made by the TAs we will employ Schoenfeld's ROG theory on decision-making. While describing how and why do teachers make their in-the-moment decisions, (Schoenfeld, 2011) asserts that what people do is a function of their resources (their knowledge, in the context of available material resources), goals (the conscious or unconscious aims they are trying to achieve), and orientations (their beliefs, values, biases, dispositions, etc). Thus if our purpose is to understand the agendas underlying the TAs' decisions we will aim to uncover each TA's orientation and goals and the resources he had at his disposal while preparing for class. Note that the decisions addressed by this theory are usually in-the-moment decisions made while the teacher is engaged in the act of teaching. We extend this framework to the teaching decisions that are made prior to the lessons, which are often planned, conscious and deliberate.

RESEARCH CONTEXT AND METHODS

This study focuses on the lessons of two TAs: TA1 is an exceptionally bright mathematics PhD student. He has been teaching TA lessons for 4 years in various courses at the time of the study and he is considered fairly popular among students. He has taught infinitesimal calculus once, 3 years prior to this study. TA2 has been the head TA of this course for more than a decade and he is a very experienced calculus teacher. He has a PhD in mathematics but he has not engaged in mathematical research for several years. The two lessons examined were the first of each of the TAs in this semester. The lesson plan was prepared by a third TA according to the demands of the lecturers and under the supervision of TA2. Note that while the TAs in this course were expected to follow the lesson plan there was no supervision during or after the TA lessons and the lecturers were usually unaware of what went on in those lessons.

The author attended the two TA lessons as a non-participant observer, audio recorded them and took notes. After each lesson the TAs described their preparation to class, and reflected on decisions made before the lecture. Each lesson was compared to the lesson plan. An initial analysis of the TAs' ROG was conducted by inferring from the TAs' actions in the classroom. Confirmation and expansion of this analysis was made in a final interview.

AN OVERVIEW OF THE TWO LESSONS

Both lessons revolved around the following definition of the derivative: Suppose f is defined in a (punctured) neighborhood of some point x_0. Then we will say that f has a derivative a at x_0 (denoted $f'(x_0)=a$) if the limit $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=a$. The first part of lesson plan was theoretical and it listed a few observations regarding the definition and the properties of the derivative (e.g. differentiability implies continuity). The second part of the lesson plan contained the following three exercises, along with concise, formal, algebraic proofs:

A. Find the derivative of the function $f(x)=\sqrt{5x+1}$ at $x_0=3$.

B. Show that the function $f(x)=\begin{cases} x^2 & x\geq 0 \\ -x^2 & x<0 \end{cases}$ is differentiable at every point and find its derivative at $x_0=0$.

C. Show that if f is differentiable at x_0 then $\lim_{h\to 0}\frac{f(x_0+h^2)-f(x_0)}{3h}=0$.

The proofs of these exercises in the lesson plan were based on the definition given above. The plan did not specify time allotment between the two parts, and did not specify any teaching/learning goals.

TA1's lesson

After introducing himself, TA1 started the lesson by writing the definition of the derivative on the blackboard. At this point TA1 stated that he would like to develop some intuition before going into examples, and initiated a teaching sequence he designed aimed at developing a geometric interpretation of the definition of the derivative. TA1 rewrote the definition in terms of epsilon and delta. Then, with a few algebraic steps, he obtained the following:

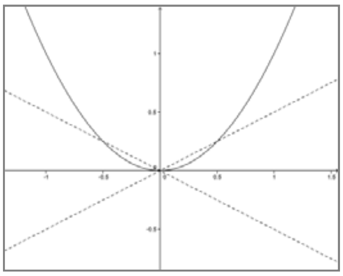
If $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=a$ then for every $\varepsilon>0$ there exists $\delta>0$ such that for every x satisfying $0<|x-x_0|<\delta$ $f(x)$ satisfies $(a-\varepsilon)(x-x_0)+f(x_0)<f(x)<(a+\varepsilon)(x-x_0)+f(x_0)$.

TA1 then clarified to the students the geometric meaning of this statement:

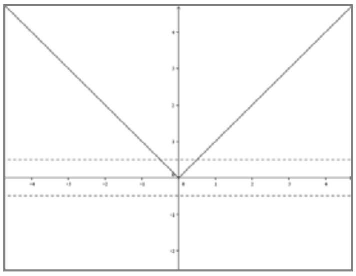
It follows from the inequality above that if f is differentiable at x_0 then $f(x)$ is bounded in some neighbourhood of x_0 from above and below by two lines with slopes $a+\varepsilon$ and $a-\varepsilon$ crossing each other at the point $f(x_0)$. We are now ready to draw a picture. At this point TA1 drew **Drawing 1** on the blackboard and repeated the explanation accompanying it with gestures toward the drawing. He noted also that from the drawing one can see that differentiability implies continuity:

> When I get closer to x_0 the graph of f gets closer and closer to the point $(x_0, f(x_0))$ (TA1's finger slides on the graph towards x_0) … I'm not giving here a full proof; I think that the fact that differentiability implies continuity is clear from this drawing.

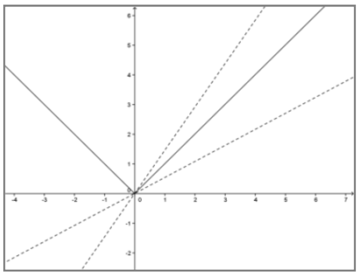
After this explanation TA1 suggested using the geometric interpretation to develop an intuition as to why the absolute value function is continuous but not differentiable at 0. For this purpose he drew **Drawing 2** and **Drawing 3**:



Drawing 1



Drawing 2



Drawing 3

After discussing with the students what can be learned through these drawings TA1 emphasized that this type of argumentation does not qualify as a proof. After that statement TA1 set aside the visual interpretation and returned to the standard definition. He wrote a concise algebraic proof showing that the absolute value function is continuous but not differentiable at zero and proceeded to exercise B of the lesson plan. Again TA1 stated that he would like first to gain some intuition. He drew the graph of the function and discussed the idea of "gluing" functions (in this case x^2 and $-x^2$) and how in some cases (e.g. the absolute value function) gluing differentiable functions yields a non-differentiable function. After this introduction TA1 presented a rigorous solution to the exercise. In order to prove the differentiability of the function he introduced the notion that the derivative is a local property of a function. TA1 emphasized to the students that the goal of this exercise is to understand this notion and he invested a great deal of time explaining it. By the time he completed the solution the lesson was over and he did not address exercises A or C.

TA2's lesson

TA2's lesson remained close to the lesson plan. Like TA1, TA2 started his lesson by writing the formal definition of the derivative. While writing the definition on the board TA2 constantly made stops to clarify the mathematical terms he used. Then

TA2 advanced quickly through the observations listed in the first part to exercise A which he wrote on the board. He then turned and asked:

> How do we start? How shall we find the derivative? Pay attention now - Whenever we learn a new concept we always start solutions by going straight to the definition.

After this introduction, TA2 wrote the appropriate limit according to the definition and simplified it to obtain: $\lim_{x\to 3}\frac{\sqrt{5x+1}-\sqrt{5\cdot 3+1}}{x-3}=\lim_{x\to 3}\frac{\sqrt{5x+1}-4}{x-3}$. He then commented:

> What kind of limit is it? (Students answer 0/0 and TA2 continues) Right! The continuity of *f* at 3 implies that that the limit is of the form 0/0. Do you all agree that f is continuous at 3? So the limit is of the form 0/0. How do we overcome this uncertainty? Do you recall what we did before in similar situations?

TA2 continued until he finally obtained the expression $\lim_{x\to 3}\frac{5(x-3)}{(x-3)(\sqrt{5x+1}+4)}$. At this point TA2 raised his voice:

> Ahha! (Pointing at the two appearances of $(x-3)$ in the nominator and the denominator) This is the "criminal" responsible for the fact that both the nominator and denominator tend to zero! We can cancel both $(x-3)$ (which are not zero by definition!) and now the limit becomes just a simple exercise, since we can use the arithmetic laws of finite limits.

After this statement TA2 completed the solution continued to exercise B. Unlike TA1, TA2 did not introduce the notion of the derivative as a local property. After completing the solution of exercise B TA2 temporarily left the course of the lesson plan. He told the students he would to give them an overview of several notions that were introduced in the lectures. After this review TA2 continued to exercise C and after that concluded his lesson.

RESOURCES, ORIENTATIONS AND GOALS

Both TA lessons were shaped by decisions made prior to the lesson. In this section we present TA1's and TA2's reflection on the reasons that led to these decisions.

TA1's decisions and ROG

The most influential decision TA1 made was to start his lesson by developing a geometric interpretation. This decision was combined with another decision - to gain a visual intuition for the two observations on the derivative prior to proving them rigorously. Another set of decisions addressed the second part of the lesson plan. TA1 chose to ignore exercise A and C and focus on exercise B. He also chose to emphasize the notion of the derivative as a local property of a function. These decisions combine with the decision is to concentrate on the theoretical aspects of the derivative rather than its applicative and procedural aspects. All these decisions were made, according to TA1, prior to the lesson.

Right after his lesson TA1 described how he prepared for class. He explained that he started by reading the lesson plan but after getting to the definition of the derivative he stopped reading and started to *Try and make sense of the definition by playing with it and drawing pictures.* This course of action led TA1 to discover a visual interpretation of the definition which he considered interesting and valuable. He then decided to present this interpretation in class and to model for the students how he himself, as an experienced mathematician, approached the definition and acted to build intuition. During the final interview TA1 reflected on his decisions:

Orientation towards the TA lessons

> The reason I opened the lesson with this interpretation is that it is not standard. It is a good example of the things I'm drawn to. It is not textual and it requires a great amount of explanation. It is not something a student would get just by reading lecture notes. I think the added value of these lessons does not lie in well-polished content but rather in the learning experience it provides for the students.
>
> I think these lessons should develop the active aspects of learning. How do you approach a task? How do you start a proof? Often before I prove theorems in class I do some preparation. I tell the students that we should first gain some intuition, try and see the big picture of the proof. Same with definitions … As mathematicians, we constantly take actions to gain intuition and students need to learn how to do that.
>
> I know there are students who prefer to learn by just writing everything that is on the board and then reading it afterwards in the comfort of their homes or before the exam. I don't claim to give very good service to these students.

Orientation towards the communication of mathematics

> Mathematicians have an overdeveloped sense of aesthetics and it makes them hide things in their proofs and present them perfectly polished. I think the students should see the process of doing mathematics in order to develop their working knowledge.

Orientation towards tasks

> (Regarding exercise A) I prefer to focus on the more difficult tasks and leave the easy tasks for the students to solve on their own, or to understand by reading the solution.
>
> (Regarding exercises B and C) I don't think students learn much from seeing me do algebraic manipulations on the board. I prefer solutions which are systematic and represent some mathematic depth, like gluing functions or the notion of a local property.

Goals for this lesson

> My overreaching goal was that the students would leave class with some intuition as to what is the derivative. The students probably heard that the derivative is the slope of the tangent line to the graph ... This is a nice intuition but there is a big gap between the tangent line which is something you can draw and the definition which express things with limits and epsilon-delta. What I tried to do is to narrow the gap.

Resources

TA1's description of the way he prepares for class gave insight to the resources available for him for designing and evaluating teaching sequences for his lessons:

> I don't read the home assignments. Usually I read the lesson plan, solve the exercises within, see what is difficult for me and try to reflect on how I deal with these difficulties.
>
> I think students should see more than one solution. For this reason I often try to prove things on my own when I read the lesson plan, and then if I get a different proof I will often present both proofs to the students.
>
> While preparing for the lesson I usually encounter something which makes me pause, maybe because it is something interesting or maybe I had difficulty justifying a certain step in some argument. I often end up taking this something to class, thinking that if I found it interesting or if it got me confused then it would probably do the same to the students. I know it is naïve to think that I and the students would find the same things interesting or confusing, but it is an inevitable working assumption.

TA2's ROG

TA2 did not make significant changes to the lesson plan and his contribution focused on his presentation. Unlike TA1, TA2 decided to advance quickly through the first theoretical part and dedicate most of the lesson to the solutions of the exercises. He paused on every term, highlighted assumptions, emphasized subtle points and generally reflected on how students should approach each task. All this represents a decision to model the behaviour of an idealized student rather than the behaviour of an experienced mathematician.

Orientation regarding the TA lessons

While discussing the goals of the TA lessons TA2 emphasized several times why he prefers to stick to content that was taught in the lessons.

> Ideally every new mathematical concept or term introduced in the lectures would be accompanied by several examples. This is not the case and one of the main goals of these lessons is to review the content taught in the lectures and to give the students an opportunity to get a firm hold on it.
>
> It is important that students will hear explanations several times, preferably from different teachers with different perspective and a different terminology. You cannot imagine how many times I have heard this "ahhh" sound of understanding in class while repeating something that was already said in the lectures.
>
> The home assignments can be very difficult. I remember myself struggling with them for hours and I think this is a good thing. However, this can be a breaking point for some of the students, especially the weaker students, and it is the responsibility of the TA lessons to support these students and to make sure they are not left behind.

Orientation regarding the role of the TAs

TA2 revealed some inner conflict between his responsibilities:

> I believe I have a responsibility towards all my students, not just those few who will later become mathematics researchers. I often tell bright students that they are "wasting their time" in my lessons because much of what I'm obligated to do is irrelevant for them.

Goals for this lesson

TA2's reflection revealed content and pedagogical goals that guided his decisions:

> Students became accustomed to differentiate functions, you start with $f(x)$ and differentiate it to obtain $f'(x)$. This approach can cause many difficulties for the students when the encounter functions which are not elementary. It was very important for me to emphasize the notion that the derivative is a property of a function at a point.
>
> One more thing to keep in mind regarding this particular lesson is that the students returned from a five weeks break (semester break). I believe that a concrete example is more effective than abstract theory for getting the students back on track. This is why it was important, in my opinion, to advance quickly to the exercises.

Resources

TA2's reflection on his preparation for this lesson reveals much about his resources:

> I think the most significant ingredient in preparation for the lesson is teaching experience. Not everything is equally difficult for the students. In every topic there are some specific difficulties and with time you learn to identify these difficulties in advance and afterwards develop various ways to address them.
>
> When I prepare for the lesson I look at the topic and think - If I was a student who does not get things very quickly, how would I see this topic, what might prove difficult for me, where will I get confused. In one topic it can be the algebra, in another topic it can be something else. I have to make a choice based on my evaluation of what the students can manage on their own, and what needs to addressed in class.

SUMMARY AND DISCUSSION

There is a well-known myth stating that excellence in mathematical teaching is just a matter of accumulated experience, clear presentation skill and sound knowledge of the subject (Alsina, 2002). One may also speculate that within the community of professional mathematicians working side by side there cannot be much variability with respect to their resources, orientation and teaching goals. In fact, this assumption seems to be institutionalized by telling the students that all the TA lessons are roughly the same.

This study not only challenges these assumptions but also highlights and analyses the differences between two instructors, both with sound mathematical background and indubitable credentials and teaching proficiency, implementing the same lesson plan. Using Schoenfeld's resources-orientations-goals theory we showed how different beliefs and attitudes their different goals and their reliance on different resources resulted in two substantially different lessons based on the same lesson plan.

A more specific contribution of this study refers to ways in which the TA's pedagogical content knowledge or lack of it played out in the importance they attached to certain teaching moves.

TA2 stated that his teaching experience is the most significant resource in his preparation for lessons, highlighting the specific difficulties he expects from students and ways of addressing them. TA2's experience told him that students will be better off with concrete examples and led him to decide to advance quickly through the theoretical part of the lesson plan. Similarly, TA2's decision to focus on the notion that the derivative is a property of a function at a point (rather than a property of a function as a whole) led him to address what he considered a common mistake, and a potential learning obstacle. In contrast, TA1's lack of calculus teaching experience forced him to choose goals according to what he found interesting or difficult led by his own introspective processes and by his aesthetics and values as a researcher trying to make sense of new concepts and ideas and designing teaching sequences accordingly. TA1 acknowledges that his perspective on the content and the ways to teach it may not necessarily coincide with the students' viewpoint.

An interesting by-product of this study the way it influenced on the TAs themselves, simply by having a chance to talk explicitly to an outside observer about their teaching. The two TAs reconsidered the beliefs and attitudes underlying their teaching. TA1 noted for example that, after reflecting on his lesson and discussing it with the author, he has decided to give more room in his class to dialogues, to listening more to his students' questions and routinely ask questions during class and thus obtaining a better understanding of their viewpoint on the content.

Paterson et al., (2011) suggested that lecturers who are research mathematicians bring different, at times conflicting, orientations into play. It is thus interesting to observe and compare the orientations of the two TAs, whose backgrounds represent these two identities. TA1 is first and foremost a researcher, and believes that it is in the best interests of his students that he will display before them the tools of the trade of a mathematician at work. At the same time TA2, who is first and fore most an instructor, believes that it is in best interest of his students that he will attend to their potential learning obstacles. TA1 directed his teaching at the mathematically oriented students (possibly future mathematicians), even though he acknowledged that by doing that he may not be providing good services to many of his students. TA2, on the other hands, felt a strong commitment to all his students and *not just those few who will later become mathematics researcher*s, but he does so at the cost of not fully attending the needs of the brighter students in his class.

While acknowledging these two identities as a potential source for conflicts regarding university math instruction it is also important to note that these two identities may also combine in fruitful ways. In this sense, it would be interesting to study a collaboration between TA1 and TA2, or any other two instructors representing their distinct identities, and examine how and when do their different orientations conflict or synergize.

We end this paper by raising an open question for the discussion within the working group regarding the desirability of pursuing uniformity (as opposed to diversity) in university mathematics teaching.

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