DEVELOPMENT AND AWARENESS OF FUNCTION UNDERSTANDING IN FIRST YEAR UNIVERSITY STUDENTS

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This article presents results from a longitudinal study on the development of first year university students' function concept, and of their awareness of this development. We used questionnaires in the first and last quarter of the first year and had 38 students participating in both tests. We found that most students' function concept did develop, but many students did not notice the development. A small number of students had low proficiency with functions but high estimation of their proficiency; these students tended to show less development in their function concept.

Keywords: function, conceptual development, self-awareness, teacher education

INTRODUCTION

Too often our students complain that their mathematical training lacks relevance for their future careers as teachers, engineers, etc. Some student teachers value the mathematical component of their study as a personal intellectual pursuit of "real mathematics" before returning to "school mathematics". We, on the other hand, see mathematics studies as an opportunity to enjoy challenging yourself with mathematical problems; to learn to value critical thinking and argumentation over dogma; and to observe your own development, as well as that of your peers. In short, to develop mathematical habits of mind, as Cuoco, Goldenberg and Mark (1996) put it.

Goulding, Hatch and Rodd (2003) investigated what British students actually carry with them from their bachelor's degrees to the teacher training (PGCE) in a retrospective study. Of the seven themes emerging from the students' responses, only two pertain to the issues mentioned in the previous paragraph; and within these categories many responses were actually negative, in that the students' views of mathematics had changed in undesirable directions (e.g., "for exams only").

In 2010, we started a project to investigate whether standard pure mathematics courses do support the development also of concepts central to "school mathematics" and applications. There are many new concepts in university mathematics from students' perspective (e.g., a focus on proof) but we instead chose a familiar concept in which the development is subtle and slow, and may go unnoticed to the students. The concept of function satisfies this criterion (Carlson, 1998) and has been indicated as one of the central topics for mathematics teacher education in the *standards* endeavor (Conference Board of the Mathematical Sciences, 2001).

We are developing a small course running alongside regular mathematics courses to help student teachers reflect on the relevance of mathematics classes for their teaching careers. To support this work we started in the fall of 2011 a longitudinal pilot study on students' development of the function concept and their self-awareness of said development. Here we report our findings from the first year of follow-up. In order to formulate our research questions we first present briefly some background material.

BACKGROUND AND RESEARCH QUESTIONS

Several studies indicate difficulties when describing mathematical concepts in different, mathematically identical, ways (see, e.g., Even, 1998; Bayazit, 2011). Although most students can visualize simple functions by drawing their graphs, they often lack ability to link the graph and the algebraic representations of the function (Vinner & Dreyfus, 1989). The intuitive notion of the function often relies on an explicit algebraic formulation (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Tall & Vinner, 1981). Functions and related notions are often treated as if they were only symbols and representations instead of proper mathematical objects (Eisenberg, 1990).

Carlson (1998) found in a cross-sectional study that even A-students take a long time to acquire solid understanding of functions: second year undergraduate students still showed many flawed conceptions. The complexity of the concept and its slow development suggest that it might be quite likely that some or even most students fail to notice their developing function concept during their mathematics studies. This is especially problematic for student teachers since they then miss out on the opportunity to reflect on the difficulty of coming to terms with functions.

As far as we have been able to find out, there is little research about students' self-awareness of long-term development of mathematical concepts (for an exception, see Bjuland, 2004). In mathematics education, a typical time frame for reflection seems to be the solving of one problem (e.g. the "looking back" stage of Schoenfeld, 1992). Following Bjuland (2004), we see both long and short time frames as necessary for self-awareness: immediate and short-term reflection is needed to notice the different aspects of the concept whereas reflection over an extended period of time is needed to compare these observations at different points in time and thus become aware of the development. The scarcity of studies on long-term reflection in mathematics education led us to consider theories of reflection and metacognition in a more general context.

A general framework for addressing issues related to reflection and metacognition was proposed by Schraw and Moshman (1995). According to them, a person's *metacognitive theory* integrates metacognitive knowledge and experiences, and can be used to explain and predict his/her cognitive behavior. They argue that one aspect in which individuals' metacognitive theories differ is the extent to which they are explicit (i.e., the extent to which one is aware of possessing such a theory).

Schraw and Moshman (1995, p. 360) point out that explicit knowledge about your own cognition makes it possible to reflect on your performance and to use this information to modify your future performance and thinking. They also seem to imply that individuals' metacognitive theories gradually develop via awareness of changes and reflection on them.

In contrast, tacit theories are developed without conscious reflection, based on personal experience or adaptations from others, and therefore it can be difficult to notice and report one's own development to others. Since an individual with a tacit metacognitive theory is not readily aware of either the theory itself or evidence that supports or refutes it, it can be very difficult to change the theory even when the theory is maladaptive and the individual is explicitly encouraged to do so. Further, the lack of conscious reflection might cause metacognitive knowledge and regulation to lack transferability between task-types. (Schraw & Moshman, 1995)

Tall and Vinner (1981) introduced a theory distinguishing between the concept image and the concept definition. Early on, this framework was applied also in the study of the function concept: it was found that a student's image of functions and his/her professed definition might be distinct and even partially contradictory (Vinner & Dreyfus, 1989). There are several ways to approach functions, each emphasizing different aspects. Therefore, one expects the function concept image to be rather complex and multifaceted.

If we look at the Tall–Vinner theory within the Schraw–Moshman framework, we find that an explicit metacognitive theory allows you to reflect on the consistency and connections between your concept definitions and concept images.

Since the development of the function concept is likely to be slow and subtle, students might not be aware of it. However, it is also possible that some students may not notice any development since there simply is none to notice. Hence we divide our research question in two:

1. Do students' function concepts develop during their first year university studies?
2. Are students aware of this development?

METHODOLOGY

Instruments

We used two questionnaires, both based mainly on Carlson's tests (1998). Our first test consisted of A1, A5, A6, A7, A14, B3 and a modification of A13 in which the linear graph was replaced by a piecewise linear one (the tasks can be found in Appendix A of Carlson's paper). The second test consisted of A2b, B2, A6, A7, and A8. The questions had been translated into Finnish and tested by P. Hästö and M. Leinonen for an earlier study (unpublished). The tests were scored following Carlson's (1998) rubrics with a few minor changes. For the modification of A13 we developed our own rubrics.

The second test also included six "self-awareness" claims which were answered on a five-point Likert scale and two other questions which are not analysed here. The claims were

- Q1. Examples of functions at the university are similar to those in high school.
- Q2. In high school it was clear to me what was meant by a function.
- Q3. It is (presently) clear to me what is meant by a function at the university.
- Q4. My understanding of functions has changed while at the university.
- Q5. I have pondered over my understanding of functions during my university studies.
- Q6. During my university studies there have appeared examples of functions contrary to my function concept.

It is, of course, clear that these questions will only provide a very sketchy picture of students' self-awareness and experience, which had to suffice for this pilot study.

Participants

The first test was administered in the second week of the first period in a class typically taken by first year students, both mathematics majors and other students with a math component (mainly physics and chemistry majors). The second test was administered in the first week of the fourth period (out of four) in an analysis class typically taken by first year majors and second or later year minors. Note that students did not follow any special courses on functions, only a standard university mathematics curriculum.

There were 98 participants in the first test and 64 in the second. Of these 38 participated in both tests. Two questions were repeated from the first to the second test; it should be noted that students got no feedback on the tests and solutions were also not distributed. A majority of these students are expected to become teachers, although the choice is not yet made.

RESULTS AND ANALYSIS

Our first observation is that students in our follow-up became significantly better at the tasks in the tests. Their scores in the two repeated tasks improved, on average, over 2.5 points (the maximum score being 5 points on each task). To compare the other tasks we use Carlson's Group 2 (A-students from a 2^{nd} year calculus course) as a reference. In the first test our students' scores were consistently (and sometimes considerably) below the reference, whereas in the second test they exceeded the reference. This strongly suggests that there was on average much improvement in students' ability to answer these kinds of questions regarding functions.

We tried to simplify and clarify the information through quantitative data reduction techniques. In particular, we looked at cross-correlations and did several test runs of principal component analyses. For instance, we looked for factors corresponding to graphical and analytic components of the function concept. Although there is a fairly clear division of the tasks into graphical and non-graphical ones, such components did not emerge from the data.

To get a view on individual improvement, we cross-tabulated the sum of scores from the two tasks included in both our tests, displayed in Table 1. In these tasks students were asked to provide examples of a function which takes integer values at non-integer points and vice versa (A6) and one which maps every real to its square except zero which is mapped to one. See Carlson (1998) for the exact formulation. From the table we see that only three students got a lower score in the second test (in fact, all dropped from 1 to 0 points), while all others improved their score, many by more than 5 points.

Table 1. Number of students with given points in Tasks A6 and A7 of the two tests.

1st test/ points	2nd test/points: 0	2	4	5–7	9	10	Total
0	1	2	2	2	0	4	11
1–3	3	0	0	3	2	11	19
5–7					1	4	5
10						3	3
Total	4	2	2	5	3	22	38

Based on the results in the two repeated tasks, we divided the students into three groups: LOW (less than 5 points in both tests), RISE (at least 5 point improvement between tests) and HIGH (at least 5 points in both tests). Three students belonged to none of these groups and are excluded from the next analysis.

For these three groups we calculated the means of the answers to the six questions mentioned on page 4. The results appear in Table 2. The answers were coded so that "strongly agree" has value 2 and "strongly disagree" –2. From the table we see that there were no radical differences between the groups. In fact, in a one-way ANOVA even the biggest between-group difference, in Question 6, was only approaching statistical significance (p = 0.085).

Table 2. Means of opinion scores. The shaded cells are discussed in more detail below.

Group	*n*	Q1	Q2	Q3	Q4	Q5	Q6
LOW	8	.3	1.0	1.0	.3	1.0	–0.4
RISE	19	.3	.3	.8	1.0	.1	–0.3
HIGH	8	.0	.9	1.4	.8	.9	–1.3
Total	35	.2	.6	1.0	.8	.5	–0.5
All	64	.2	.6	1.1	.8	.5	–0.4

Nevertheless, we combine indications from several answers to form a tentative conclusion, or, if you will, a hypothesis for further study. Q4 (change of understanding) is consistent with Q2 (high school) and Q3 (university) in that groups RISE and HIGH now rate their understanding higher than in high school, and correspondingly rate it as having changed more that group LOW.

Group HIGH has consistently higher view of their function understanding than group RISE, which is consistent with test scores. However, students in group LOW find that they have had a fairly clear understanding of functions from high school, which has not changed. It appears that these students have an erroneous and overly optimistic view of the adequacy of their function concept.

Question 5 (pondering) is somewhat of a dilemma: the group with the least pondering (RISE) has improved most drastically. Finally, Q6 (new examples) is consistent with the groups in that the consistently high-scoring individuals found fewer examples which did not fit into their conception of function. Surprisingly, across all students there was a tendency to disagree with the statement that they had encountered new types of functions at the university. This may be due to the inclusion of the technical term “function concept” in the question.

Qualitative analysis

Based on students’ responses to questions A5–7, we identified four students as major improvers and did a more thorough data-driven analysis of their answers. In the analysis we marked all interesting responses; since the data set was not large, there was no need for categorization or data reduction and all marked responses are presented.

Our first observation in the qualitative analysis of the two questionnaires was the obvious improvement in students’ understanding of “the language of functions”, as Carlson (1998) puts it. Initially the students demonstrated great difficulties in translating a verbal description of a function into algebraic notation: e.g., when giving an example of a function all of whose values are equal (task A5), one of them wrote “for example, $f(x): \quad x = 2$”, and another one started to solve the equation $x - y = y - x$, and after a couple of steps concluded that $x = y$. Apparently these students understood neither what is meant by “the value of a function” nor the usual notation by which functions are defined. However, three of these students showed very little difficulty in algebraic manipulation while expressing the diameter of a circle as function of its area (the fourth one left it blank), but all of them neglected to sketch the graph.

It also seems that these students thought that all functions must be defined by a single algebraic formula. When asked (task A7) to give an example of a function which assigns to every number different from 0 its square and to 0 assigns 1, two of the students simply wrote “x^2” while two left it blank. In the second part none had difficulty in defining functions in a piece-wise manner.

Although there is little doubt about the students' improved writing in a formal mathematical style, there is some doubt about the depth of this improvement. For example, in the challenging task B2, one of the four students started the solution by defining $f(x) = x^2$ and then $f(y) = y^2$, as though the second did not follow from the first (a misconception also documented, e.g., by Sajka, 2003).

The difficulty in interpreting functional information from a given graph (rather than interpreting it literally) appears to be somewhat persistent. In Task A8 (second test) the students were given a graph illustrating the speed of two cars, A and B, as a function of time (see Fig. 1). All the four students demonstrated very little difficulty in interpreting static graphical information (speed) or relatively standard dynamic information (acceleration), yet all of them failed in Task A8d, in which they were asked to describe the relative position of the two cars over a time interval. Although they noticed that the car A was driving faster than B for the whole time, they still concluded that B was catching up with A because B had accelerated so much whereas A had driven with nearly constant speed.

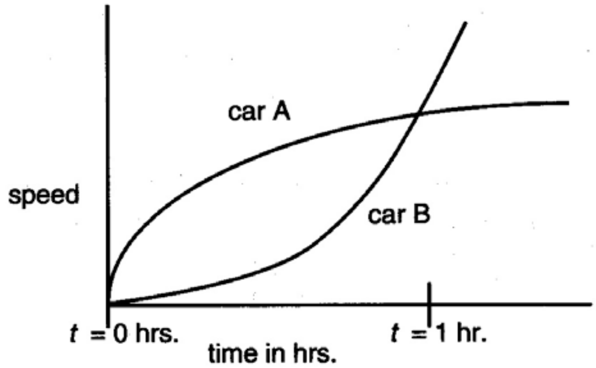

Figure 1. The graph from Task A8.

DISCUSSION

In answer to our first research question, we found that the function concept of most students does seem to improve. In particular, courses in pure mathematics then also support the education of student teachers and applied mathematicians. The answer to the second question, regarding students' awareness, is more complicated. Let us elaborate on this.

Our results are consistent with Carlson's study (1998). In our first test students had difficulties very similar to those in Carlson's Group 1 (A-students from a college algebra course): they had problems understanding the language of functions, for instance when defining functions piecewise. In the second test, however, our students seemed to have the same kind of difficulties as Group 2 (A-students from a 2^{nd} year calculus course), e.g., interpreting dynamic functional information over an interval. Interestingly, although many participants in our first test apparently thought that a function must be defined via a single algebraic formula, this was not the case in the second test; in contrast, Carlson's Group 2 subjects had this misconception. We interpret this as meaning that our students were slightly more advanced in their general understanding of functions than Carlson's Group 2 students.

Between the two tests most of the students had taken introductory courses on both abstract and linear algebra, and advanced calculus. In these courses functions appear mainly through definitions and further properties (bijectivity, linearity, etc.) whereas little explicit focus is placed on development of intuitions (i.e. the construction of concept images). Nevertheless, the quantitative analysis showed that students made great progress in function tasks. This means that students were able to improve their function language without explicit instruction. Unfortunately, we cannot say how much of the improved scores is due to conceptual change and how much is a consequence of adopting the formal language used at university.

The qualitative analysis showed that this progress was tenuous in the sense that there was a tendency to relapse into old ideas when facing non-standard tasks with high cognitive demand. Apparently, students had least difficulty in tasks requiring algebraic manipulation of quantities. This is probably due to the Finnish curriculum in which a considerable attention is paid to solving equations via algebraic manipulation. On the other hand, in many high schools there is passing mention also of more abstract notions of functions, so the best students may have picked it up there. This may explain why they did not find examples at the university to be beyond their function concept. Altogether, these observations are consistent with Carlson's (1998) finding that students do not use the newest tools in their conceptual arsenal proficiently.

Our tests did not uncover a graphical and analytical component in the sense that there would have been greater correlation within the groups of graphical and non-graphical tasks. This may be due to floor and ceiling effects in the task scores which led to skewed distributions; we found several nonlinear relationships between tasks where only students with 5/5 points on problem X got non-zero points on problem Y. This means that the difficulty range of the problems was not optimal. On the other hand, it seems that also in most other studies the graphical and analytical components are postulated rather than recovered from the data. Our observations of graphical components are limited to the qualitative analysis. Here our findings mirror Bayazit's (2011) results of students' limited abilities to shift from a graphical expression to algebraic expression, with most depending on algebraic expressions.

Some answers indicate that the concept of variable is also problematic. For instance, one student introduced a function by defining it for several variables ($f(x)=x^2$ and $f(y)=y^2$) instead of just once. We do not know to what extent our students' problems are due to deficiencies in such prerequisite concepts as that of the variable.

About a quarter of the participants performed poorly and did not improve their performance between the first and second test. Interestingly, these students on average had a high estimation of their understanding of functions — in fact, they estimated that the function concept had been clear in high school and had remained clear since then. If they do not come to terms with the need to develop their function concept, this might cause serious problems for their studies and possible teaching careers. From course grades and other feedback these students must know that their studies are not progressing as well as they should be. Therefore they seem to be misattributing this lack of progress to some other factors than a weak function concept. This suggests that they do not have effective means of checking their metacognitive theories against reality, which indicates a persistent tacit metacognitive theory.

A second group, consisting of more than half of the participants, improved between the tests. Moreover, this group realistically estimated their function concept as having been average and then having improved. Unfortunately, we only asked participants to rate the clarity of the concept in the second test. Thus we have only a retrospective estimation of what the students thought after high school. Do they rate the conceptual clarity as low because they now understand it better? Or is it the case that they were unsatisfied with their understanding at the beginning of the studies, and were thus more open to new influences? This remains a question for further study.

If the retrospective rating is accurate, then it might allow us to design a diagnostic test for the beginning of studies which detects students at risk of belonging to the LOW category as those with low performance and high estimation of their function concept. Alcock and Simpson (2004) found that this kind of profile (low performance and high confidence) is typical for some graphically oriented students. Whether there is such a link in our case also remains to be determined. Another open question is whether this profile is context specific, or whether the students follow the same path in all areas of mathematics.

The results of the first year of our study in general agreed with our expectations. Students' concepts improved and most said as much in the questionnaire. Whether this group contains a subgroup of improvers who do not notice the change is a question for future study. Almost a quarter of the students did not improve, and showed low self-awareness of their situation. This pilot study had both deficiencies and strengths. In future versions we will diversify the data collection to get a better picture of the groups tentatively identified in the analysis above.

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