A CROSS-NATIONAL STANDARDS ANALYSIS: QUADRATIC EQUATIONS AND FUNCTIONS

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With the advent of the Common Core State Standards (CCSS), more researchers are eager to learn the features of the new standards and concerned about how the CCSS can impact their work. This research study will compare the characteristics of the CCSS for Mathematics (CCSSM) with an array of international written learning expectations for mathematics. We designed a cross-national study to examine how the topic of quadratic equations and functions is introduced in four different countries: the Caribbean, China, Turkey, and the U.S. These standards were analyzed under three dimensions of characteristics: content, mathematical reasoning, and cognitive level. The results show that all the standards introduce the foundational concepts of quadratic functions, however, with various procedural and conceptual expectations.

INTRODUCTION

The newly released *Common Core State Standards (CCSS*) have aroused wide interests in the field of education. As these standards state,

> The standards are designed to be robust and relevant to the real world, reflecting the knowledge and skills that our young people need for success in college and careers. With American students fully prepared for the future, our communities will be best positioned to compete successfully in the global economy. (Common Core State Standards Initiative [CCSSI], 2010)

How to make a smooth transition to this new set of standards has become a salient question that concerns K-12 teachers, researchers, policy makers, students and parents. These standards also have implications for other countries as well, since international students and scholars are an increasing part of the post-secondary education population in the United States. As a group of international researchers studying mathematics education in the United States (authors of this paper are from the Caribbean, China, and Turkey), we set out to investigate *CCSSM* from an international perspective using the standards and teaching plans that we are most familiar with as benchmarks. We designed a cross-national comparative study to investigate the similarities and the differences of learning expectations for students among these four economically, socially, culturally, and geographically diverse countries. These countries also vary in terms of the structure and history of their educational systems, as well as the implementation of their educational policies and intended curriculum. For instance, the Caribbean, China, Turkey, and the U.S. have various written formats to communicate what and how their students are expected to learn (these differences will be discussed subsequently), thus for simplicity we will refer to all of these documents as *Written Learning Expectations for Mathematics* (*WLEMs*). At the same time, we recognize that the *CCSSM* is an emerging influence on the education landscape of the U.S. and the world.

In this study, we focus our cross-national comparison on the *WLEMs* related to quadratic equations and functions in these four countries regardless of the grade level. As it has been stated by other researchers, our ultimate goal of doing this cross-national comparison is not to simply rank nations, but to ascertain the strengths and weaknesses of different educational systems and to also provide a basis for considering how to improve the teaching and learning of mathematics (e.g., Cai, 2001; Porter & Gamoran, 2002; Robitaille & Travers, 1992).

BACKGROUND

Standards Analysis in Previous Studies

A few studies have focused on curriculum standards analysis; however, most of them only look at a single aspect of these standards, such as the coherence (e.g., Schmidt, Wang, & McKnight, 2005), format (e.g., Reys, Dingman, Nevels, & Teuscher, 2007), or content (e.g., Reys, 2006).

Schmidt et al.'s (2005) examination of the coherence of mathematics and science standards revealed that, compared to the six high-achieving countries in the *Trends in International Mathematics and Science Study (TIMSS)*, each grade in the various U.S. national standards devoted instructional attention to many more topics and these topics stayed in the curriculum for more grades. Reys et al.'s (2007) report summarized the format of state-level curriculum standards and graduation requirements for high school mathematics. The result showed that states are varied with respect to required mathematics credit hours and courses for graduation. In work conducted by the Center for the Study of Mathematics Curriculum (CSMC), Reys (2006) reported results of comparing state-level K-8 mathematics curriculum-standards documents in three content strands: number and operation, algebra, and reasoning. The report highlighted how different topics within these strands were sequenced and emphasized across the different state documents.

Only one research study examined the multiple aspects of the standards: clarity, content, mathematical reasoning, and negative qualities (Raimi & Braden, 1998). Because this study aimed to critique the standards, not all the categories were suitable for international comparison.

Significance of Quadratic Equations and Functions

Solving quadratic equations is one of the most conceptually difficult topics in the secondary school mathematics curriculum (e.g., Vaiyavutjamai & Clements, 2006), and educators should be aware of these difficulties and be prepared to help students to confront these challenges. Even though quadratic equations play an important role in secondary school curriculum around the world, studies concerning teaching and learning quadratic equations are quite rare in algebra education research (Kieran, 2007; Vaiyavutjamai & Clements, 2006).

In addition, although quadratic functions are one of the most important concepts extending beyond linear functions in the secondary school mathematics curriculum, students have struggled to understand quadratic functions (Ellis & Grinstead, 2008). Students' struggles with quadratic functions are based on a few key areas, including inflexible connections between multiple representations; rigid view of graphs as whole objects; incorrect interpretation of the role of parameters; and tendencies to incorrectly generalize from linear functions (Ellis & Grinstead, 2008). These findings indicate that educators should be aware that students must move flexibly and develop connections among different representations in order to establish a functional understanding.

METHODOLOGY

Theoretical Framework

The process of building a robust framework within which to perform a cross-national comparison led us to a three-dimensional framework: content, mathematical reasoning, and cognitive demand. For the content category we chose to follow the definition of content as "coverage," and our analysis involved a comparison of the topics and concepts that each set of standards requires in relation to quadratic equations and functions.

The second dimension of the analytical framework is mathematical reasoning in relation to quadratic equations and functions. This part of our framework is based on the recommendations of NCTM's (2009) Reasoning and Sense Making document. First of all, the document suggests *Reasoned Solving* with algebraic symbols, which means that "problem solving with equations should include careful attention to increasingly difficult problems that span the border between arithmetic and algebra" (p.34). This ensures that students can see that algebra extends arithmetic reasoning and is a more powerful approach when solving more challenging problems alone.

In terms of functions, the document suggests three essential elements:

- *Using multiple representations of functions.* Representing functions in various ways, including tabular, graphic, symbolic (explicit and recursive), visual, and verbal; making decisions about which representations are most helpful in problem-solving circumstances; and moving among those representations.
- *Modeling by using families of functions.* Working to develop a reasonable mathematical model for a particular contextual situation by applying knowledge of the characteristic behaviours of different families of functions.

- *Analyzing the effects of parameters.* Using a general representation of a function in a given family (e.g., the vertex form of a quadratic, $f(x) = a(x - h)^2 + k$ to analyze the effects of varying coefficients or other parameters); converting between different forms of functions (e.g., the standard form of a quadratic and its factored form) according to the requirements of the problem-solving situation (e.g., finding the vertex of a quadratic or its zeros) (p. 37).

The third dimension, cognitive level, focuses on how students are expected to learn concepts, what they are expected to do in the process, and how they are to demonstrate their understanding. We chose to use the Bloom's Taxonomy which is a classification of levels of learning objectives that educators set for students. It was initially proposed by a committee of educational psychologists chaired by Benjamin Bloom in 1956. In this paper, we used a revised version of the taxonomy created by Anderson and Krathwohl (2001). Bloom's Taxonomy divided educational objectives into three "domains": *Cognitive, Affective,* and *Psychomotor*. Within these domains, learning at the higher levels depends on having attained prerequisite knowledge and skills at lower levels. In this study, we only attend to skills in the cognitive domain. There are six levels in the taxonomy, moving from the lowest order processes to the highest: (a) *knowledge*—Exhibit memory of previously-learned materials by recalling specifics and universals, methods and procedures, and patterns, structures, or settings; (b) *understand*—Understand facts and ideas by organizing, comparing, translating, interpreting, giving descriptions, and stating main ideas; (c) *apply*— Solve problems in new situations by applying acquired knowledge, facts, techniques, and rules in a different way; (d) *analyze*—Examine and break information into parts by identifying motives or causes. Make inferences and find evidence to support generalizations; (e) *evaluate*—Present and defend opinions by making judgments about information, validity of ideas, or quality of work based on a set of criteria; and (f) c*reate*—Create new product or point of view (Krathwohl, 2002). We also referenced a list of action words of the taxonomy categorized by other scholars (Center for University Teaching, Learning, and Assessment [CUTLA], n.d.).

Data Collection & Analysis

The quadratic functions and equations *WLEMs* from all the different countries in the study were collected for anlysis, and the Turkish and Chinese *WLEMs* were translated by the members of the research team from these respective countries. For the content dimension team members recorded all the distinct topics and concepts related to quadratic equations and functions. For the mathematical reasoning dimension each country's *WLEMs* were coded for unique opportunities for mathematical reasoning according to the framework criteria. A narrative was compiled for each country and a cross-national analysis was conducted. For the cognitive level dimension of the framework, we used Bloom's Taxonomy action verb analysis to code each *WLEM* into

Bloom's categories. During the data analysis process, we wanted to make sure that the number of verbs that are used represented the actual number of tasks in each *WLEM* (a *WLEM* could include multiple tasks), and this rule was applied to all of the countries. To do this, we coded each *WLEM* based on the verbs that dictated its learning goals. Each set of *WLEMs* was coded by two researchers. Discrepancies were discussed and resolved by the research team.

DATA INTRODUCTION

Caribbean *WLEMs*: The most recent Caribbean *WLEMs* document was written in 2008 and is 47 pages long. It provides a rationale for the mathematics included, general and specific course objectives, a summary of how the syllabus is organized, and detailed descriptions of the exams that would be used to assess students (including the distribution of content to be covered and the details of the topics covered). The mathematical objectives listed in the *WLEMs* for quadratic functions list specific skills that students should acquire and gives a few example problems. In the general objectives, students are given esoteric mathematical expectations that are very broad. Some examples of the general learning expectations include: learning to treat algebra as a language and a way of communicating, appreciating the role of symbols and algebra techniques in problem solving, and being able to reason abstractly (*Caribbean Examination Council* (CXC) syllabus, 2008).

***CCSSM*:** The *CCSSM WLEMs* is a 93 pages long document. It consists of two main sections: Standards for Mathematical Practice and Standards for Mathematical Content. In this study, we focus on the content section, which is organized by grades (K-8) and organized by content strand at the high school level (Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability). It provides descriptions of what students should know and provides one or two example equations or problems where appropriate.

Chinese *WLEMs*: The *Full-time Compulsory Education Mathematics Curriculum Standards (FCEMCS)* was enacted by the Ministry of Education of the People's Republic of China, and published by Beijing Normal University Press in 2001. It is a 44 pages long document which includes three stages: *stage one* (grades 1-3), *stage two* (grades 4-6), and *stage three* (grades 7-9); four areas of mathematics content: numbers and algebra, figures and geometry, statistics and probability, and integration and practice; and four types of objectives: knowledge and skills, mathematical thinking, problem solving, and emotion and attitudes.

The latest edition of the *General High School Mathematics Curriculum Standards (GHSMCS)* was published by People's Education Press in 2011. This high school mathematics curriculum consists of two types of courses: compulsory and elective. The compulsory course contains five modules: *Math 1-5*. For the high school standards, we only focus on the module *Math 1*which is a 5 pages long document and covers the *WLEMs* in relation to quadratic equations and functions

Turkish *WLEMs*: The high school mathematics curriculum (grades 9-12) has two main areas: mathematics and geometry. The mathematics curriculum is 364 pages long, and the geometry curriculum has 207 pages for grades 9 and 10, 127 pages for grade 11, and 176 pages for grade 12. In the *WLEMs*, every concept was designed according to four sub-sections: *subfields*, *gains*, *hints for the activity*, and *explanations*. These sub-sections include gains that what students learn about the context at the end of the instruction, examples that are helpful for teachers to teach a concept, and explanations that are recommendations for teachers to use in their instructions. These standards are the requirements to be achieved by all students, and students must learn those standards in order to go a grade further.

FINDINGS

Content

In terms of content for quadratic functions and equations, the Caribbean *WLEMs* provided an introduction to basic principles, skills, and processes. Students are expected to be able to factor quadratic expressions, solve abstract quadratic equations and context specific ones in word problems, use symbols to denote quadratic functions, and be able to graph and interpret quadratic functions. Two optional specific objectives require that students find and understand the significance of the axis of symmetry and find the number of roots for a particular quadratic equation. The content range among *CCSSM*, Chinese and Turkish *WLEMs* are similar to those of the Caribbean, but there are also notable differences. For *CCSSM*, the content on quadratic functions requires that students solve quadratic equations using different methods, using multiple representations (graphical, tabular, and symbolic), comparing and evaluating the characteristics of quadratic functions with other families of functions; and modelling real or natural patterns with quadratic functions and also problem-solving by using quadratic functions in realistic contexts. *CCSSM* also emphasizes the relationship between different types of functions. The Chinese and Turkish *WLEMs* also include fundamental properties of quadratic equations and functions (e.g., zero roots and extreme values) and basic problem solving means (e.g., graphing, factoring, and completing-square method). Additionally, there are some minor differences: Chinese *WLEMs* are more specific on the properties of functions (e.g., direction of the opening, monotonicity, domain and range, and even/odd functions) whereas Turkish *WLEMs* also embed their mathematical content in specific examples and have made problem solving a foundational part of the *WLEMs*.

Mathematical Reasoning

Reasoned Solving of equations. The Caribbean *WLEMs* clearly require students to be able to solve quadratic equations and solve a pair of equations with one of them is non-linear equation. The *CCSSM* high school algebra standards require students to create quadratic functions that describe numbers and relationships, to solve quadratic equations in one variable, and to understand solving equations as a process of reasoning. Both *CCSSM* and Chinese *WLEMs* emphasize the connection between equations and functions; that equations can be solved by using graphs of the corresponding functions. The Turkish *WLEMs* emphasize the connections between the roots and coefficients of quadratic equations.

Using multiple representations of functions. The Caribbean and Turkish *WLEMs* have similar requirements with regard to the form of representations. They both expect students to use graphical, symbolic, and tabular representations while the use of technology is not mentioned. The *CCSSM* are the only *WLEMs* among the four countries that mention the verbal representation. They also state that students should be able to graph functions both by hand and with technology. The Chinese *WLEMs* emphasize plotting the graphs of quadratic functions and suggest using technology, such as the calculator, to analyze the graph and to find the solution of the corresponding equation.

Modelling by using families of functions. There is no explicit mention of modelling with quadratic functions in the Caribbean *WLEMs*, except for one general objective stating that students should "appreciate the usefulness of concepts in relations, functions and graphs to solve real-world problems" (*CXC*, 2008 p. 24). The *CCSSM* has a set of modelling *WLEMs* which are embedded in other *WLEMs* and appear throughout the entire high school section. For example, there is an emphasis on modelling skills in the domain of building functions, which requires the ability to describe a relationship between two quantities in a contextual situation (F-BF). The Chinese *WLEMs* not only require a deep understanding of the concept of modelling through various examples, but also require the ability to represent the function with an appropriate form (e.g. graph, table, and equation) in a real world context. The Turkish *WLEMs* do not have specific standards on modelling. They provide many examples, but in these examples, the model is usually given.

Analyzing the effects of parameters. The Caribbean and Chinese *WLEMs* include understanding and using the form $f(x) = a(x - h)^2 + k$, but this *WLEMs* is optional for Caribbean students. All of the *WLEMs* require a general representation of the quadratic function except for the *CCSSM*. However, the *CCSSM* mentioned using various equivalent forms. Uniquely, they require representing the quadratic equation and its solution in the form of complex numbers $a \pm bi$ for real numbers a and b. The Turkish *WLEMs* require the use of the form $f(x) = ax^2 + bx + c$. All the *WLEMs* suggest studying the properties of functions through graphs (e.g., finding the zeros, symmetry of the graph, intercepts and extreme values) while they have slightly different emphases on problem solving methods. The Caribbean *WLEMs* focused on analyzing the graph (the symbolic way is optional); the *CCSSM* focused on factoring and completing the square; the Chinese *WLEMs* suggest various methods, such as factoring and completing the square, root-formula, and dichotomy (by using the calculator); the Turkish standards do not have specific requirements for problem solving methods.

Cognitive Level (Bloom's Taxonomy)

Based on the cognitive level dimension, we analyzed 45 *WLEMs* of quadratic equations and functions across all the countries and we found 72 tasks in those *WLEMs* (See Table 1). The results of the cognitive-level analysis show that the Bloom's *apply* category is most common across all the *WLEMs*; 46 tasks were identified in this category. In the Caribbean *WLEMs* 100% of the action words represent the *apply* category. *CCSSM* have 14 tasks, the Chinese *WLEMs* have 11 tasks, and Turkish *WLEMs* have 11 tasks in the *apply* category.

We also found some distinctive characteristics among the *WLEMs*. As mentioned above, the Caribbean *WLEMs* have only one category of action words, *apply*. The *CCSSM* do not have three categories of action words: *knowledge, analyze,* and *create*. The Chinese *WLEMs* have the broadest coverage of the cognitive levels; they include all the categories except for the *create* level, and they are the only standards which reach the level of *analyze.* Additionally, the Chinese *WLEMs* emphasize the foundations of *knowledge* and *understanding*. Turkish *WLEMs* do not have any action words in the levels of *analyze* and *evaluate* but they are the only *WLEM*s which include the level of *create*.

	Knowledge	Understand	Apply	Analyze	Evaluate	Create	Total
CCSSM	0	5 extent, interpret, estimate, observe	14 choose, complete the square, factor, graph, solve, write, show, sketch, relate, calculate, determine, combine	0	1 properties	0	20
Caribbean	0	0	10 determine, draw, factorize, interpret, sketch, solve, use	0	0	0	10
Chinese	7 know	7 determine, know, obtain, represent, understand, learn	11 apply, draw, examine, find, plot, represent, solve, use	2 experience	1 investigate	0	28
Turkish	1 identify	1 identify	11 calculate, find, show, sketch, write	0	0	1 organize	14

Table 1: Action Words for Bloom's Taxonomy in Cross National Comparisons

DISCUSSION

One of the limitations of this study was the different representations of learning objectives used in the different *WLEMs* of the countries studied for this report. While there were somewhat similar content coverage goals, what was expected of students, how the content emerged, and the role of the teacher in deciding when the content was to be taught varied across the *WLEMs*. We encountered several issues while coding the *WLEMs* translated from Turkish and Chinese. There is also the possibility that some of the original intent of the *WLEMs* were lost in translation. Besides identifying the action verb using Bloom's taxonomy we also had to read the each WLEM completely to determine whether the verb was categorized correctly or if the verb needed to be placed into a different Bloom's level depending on the requirements from the *WLEMs*. The conclusions that we can make about these four different *WLEMs* are only with regard to quadratic functions and equations. Further studies would be needed to make larger comparisons regarding all of the content taught in high school mathematics.

Finally, the *WLEM*s still do not give an adequate picture of the opportunities that students have to learn about quadratic equations and functions. We cannot claim to understand the quality of instruction or to predict the level of achievement that students who benefit from these *WLEMs* experience in their post-secondary careers. However, we can conclude that the learning expectations for these concepts are comparable across the four countries they are diverse in many aspects.

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