PERSISTENCE AND SELF-EFFICACY IN PROOF CONSTRUCTION

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We first discuss our perspective and three useful actions in proof construction that depend on persistence. Persistence is important for successful proving because it allows one to "explore", including making arguments in directions of unknown value, until one ultimately makes progress. Persistence can be supported by a self-efficacy belief, which is "a person's belief in his or her ability to succeed in a particular situation" (Bandura, 1995). We discuss a study of U.K. undergraduates' perceived sense of self-efficacy with regard to proving (Iannone & Inglis, 2010). We then examine actions needed for a successful proof construction of a theorem given to mid-level U.S. undergraduates in a transition-to-proof course. We contrast those actions with the actual actions of a mathematician proving the same theorem.

Key words: proof construction, persistence, self-efficacy, undergraduates, mathematicians.

INTRODUCTION

In this paper we first discuss our perspective on proof construction. Then, drawing on observations from a multi-year teaching experiment, point out three aspects of proof construction that appear to be especially difficult to teach. We suggest that the teaching difficulty arises from a need for students to have a kind of persistence, which in turn may depend on students' sense of self-efficacy. After discussing self-efficacy, we consider a prior study of U.K. undergraduates' performance and perceived sense of self-efficacy with regard to proving(Iannone & Inglis, 2010). We next illustrate the way that self-efficacy and persistence are valuable by discussing the proof construction of a specific theorem that students in the teaching experiment are asked to construct and how one mathematician approached proving it. Then, after a brief discussion, we end with some teaching implications.

OUR PERSPECTIVE

We view constructing a proof as a sequence of actions (Selden, McKee, & Selden, 2010). Some of these are physical, such as writing a line of the proof, and some are mental, such as focusing on the conclusion or "unpacking" its meaning. Such actions are taken in response to certain kinds of situations in a partly constructed proof. With practice, the links between some repeatedly occurring proving situations and the resultant actions will become automated, thus reducing the burden on working memory in future proof construction. Many such actions taken during the construction of a proof are not recorded in the final written proof. Thus, it may be difficult to mimic a given proof, or even parts of it, when constructing another proof.

Three useful actions in proof construction

In several iterations of teaching a U.S. second-year university transition-to-proof course in a modified Moore Method way (Coppin, Mahavier, May, & Parker, 2009; Mahavier,1999), we have observed the following three useful actions that can be undertaken in proof construction situations. *(1.) Exploring.* In constructing part of a proof, one may understand both what is to be proved and what is available to use without having any idea of how to proceed. In such situations, one might reasonably try to prove something new of unknown value. However, we suspect many students are reluctant to do this, perhaps lacking confidence in their own ability to use whatever new they might prove. *(2.) Reworking an argument in the case of a suspected error or wrong direction.* In constructing a proof, one may come to suspect one has made an error or is arguing in an unhelpful direction. An appropriate response would be to rework part of the argument. However, we suspect many students are reluctant to do this, perhaps because they lack confidence in their own abilities to produce something new and better than before. *(3.) Validating a completed proof.* Upon completing a proof, one should read it carefully for correctness from the top down, checking whether each line follows from what has been said above. We suspect that few students do this, perhaps because they do not think that they are able to find errors in their own, just completed, proofs.

Some student errors may depend on a wrong belief about mathematics or logic or on a misinterpretation of a definition. Such errors can be pointed out and an explanation can be provided by a teacher. However, the above three actions in proof construction are not about correcting an error, but about habitually acting appropriately in particular situations. They seem to depend on students' views of their own abilities, that is, on a sense of self-efficacy and persistence. We suspect that encouraging this kind of appropriate behaviour may require some kind of teaching beyond explaining errors.

SELF-EFFICACY

Self-efficacy is "a person's belief in his or her ability to succeed in a particular situation" (Bandura, 1995). Of developing a sense of self-efficacy, Bandura (1994) stated that "The most effective way of developing a strong sense of self-efficacy is through mastery experiences," that performing a task successfully strengthens one's sense of self-efficacy. Also, according to Bandura, "Seeing people similar to oneself succeed by sustained effort raises observers' beliefs that they too possess the capabilities to master comparable activities to succeed."

According to Bandura (1994), individuals with a strong sense of self-efficacy: (1) view challenging problems as tasks to be mastered; (2) develop deeper interest in the activities in which they participate; (3) form a stronger sense of commitment to their interests and activities; and (4) recover quickly from setbacks and disappointments. In contrast, people with a weak sense of self-efficacy: (1) avoid challenging tasks;

(2) believe that difficult tasks and situations are beyond their capabilities; (3) focus on personal failings and negative outcomes; and (4) quickly lose confidence in personal abilities.

Bandura's ideas "ring true" with our past experiences as mathematicians teaching courses by the Moore Method (Coppin, Mahavier, May, & Parker, 2009; Mahavier,1999). Typical Moore Method (advanced undergraduate or graduate) courses are taught from a brief set of notes consisting of definitions, a few requests for examples, statements of major results, and those lesser results needed to prove the major ones. Exercises of the sort found in most textbooks are largely omitted. In class meetings, the professor invites individual students to present their original proofs and then very briefly comments on errors. Students are typically forbidden to read anything on the topic or to discuss it with anyone other than the professor. Once students are able to successfully prove the first few theorems, they often progress very rapidly in their proving ability, even without apparent explicit teaching, and even when subsequent proofs are more complex. Why should this be? We conjectured then, and also now, that students obtained a sense of self-efficacy from having proved the first few theorems successfully, and this helped them persist in explorations and re-examinations needed to prove subsequent theorems. That is, they learned to gather as much information as they could and explored various ideas, whether or not they could initially "see" their usefulness.

Similarly, in discussing geometry conjectures, de Villiers (2012) pointed out places where a novice might

> lose hope of getting anywhere as it's not obvious from the start this will lead somewhere useful. However, students should be encouraged to persist … and not so easily give up … One might say that a distinctive characteristic of good mathematical problem solvers [and provers] are that they are 'stubborn', and willing to spend a long time attacking a problem from different vantage points, and not easily surrendering. (p. 8)

However, dogged persistence in a *single* direction, without accompanying monitoring and control, can lead to what Schoenfeld (1985, p. 316) called mathematical "wild goose chases."

Further, the American mathematician Steven Krantz (2012) has written, in his book on mathematical maturity, that

> at least as important [as being smart] is perseverance or tenacity. Mathematics can easily be discouraging. … you may find that you are stuck for a goodly period of time. … You either do not know enough, or do not know the right things. This is why tenacity is important. You must have adequate faith in yourself [sense of self-efficacy] to know that you can battle your way through the problems. … Not unrelated to the idea of tenacity is the property of being comfortable with delayed gratification. … Once you are challenged to generate your own proofs and counterexamples, you are frequently at odds, and often frustrated. … The unifying theme for dealing with the need for tenacity and the need to

> deal with delayed gratification is self-confidence [sense of self-efficacy]. You need to believe in your own abilities, and you need to believe that *you can actually do this work*. (pp. 97-99).

Thus, it would seem that a sense of self-efficacy, that is, a belief in one's ability to succeed on a particular kind of task, enables one to persist despite frustrations or wrong paths and that this is an important part of doing mathematics, and in particular, of proving original results. Indeed, we suspect that it is an important part of much creative cognition in general.

UNDERGRADUATES' SENSE OF SELF-EFFICACY AND PERFORMANCE IN PROOF CONSTRUCTION

One study investigated the relationship between U.K. undergraduates' perceived sense of self-efficacy with regard to constructing proofs and their actual performance on proof tasks (Iannone & Inglis, 2010). The study consisted of a two-part questionnaire administered to 76 first-year undergraduates studying mathematics (or studying for a joint degree with a substantial mathematics component) at a highly ranked U.K. university. The data were collected in the first semester after the students had completed eight weeks of degree level mathematics study. The first part of the questionnaire consisted of 28 statements, such as "I am good at writing mathematical proofs" and "I never know how to start a mathematical proof", that students decided were, or were not, characteristic of them (using a Likert scale). These statements were designed using the self-efficacy literature of Bandura and others. The second part of the questionnaire consisted of four novel proof construction tasks that the researchers and a mathematics lecturer considered feasible for that cohort. For example, one proof task was: *Let d, a, and b be integers. Prove that if* $d \mid a$ *and* $d \mid b$ *then* $d^2 \mid (a^2 + b^2)$. The result was that "those participants who had low perceived proof self-efficacy [scores] tended to do worse on proof construction tasks than those participants who had a high perceived proof self-efficacy [score]." (p. 5). The authors concluded that their data "confirms that there is a positive correlation between students' perception of their abilities at producing proofs and their actual proof production performance, in line with general literature about self-efficacy and mathematical ability." (p. 5)

While this study is suggestive, it did not indicate causality or consider a mechanism that would relate a perceived sense of self-efficacy with success at proof production. Indeed, because the proofs constructed by the undergraduates were not too demanding and were to be completed within twenty minutes, this study by Iannone and Inglis (2010) sheds little light on any causality relationship between self-efficacy and persistence in constructing more difficult proofs, where noticing and correcting errors and pursuing exploratory argument directions not known in advance to be helpful, would come into play. For this purpose, in the next section, we provide a hypothetical proof construction of a more difficult theorem and suggest where students might need to reflect on and explore various argument directions, in particular, directions that probably could not be seen to be helpful before they were undertaken.

A HYPOTHETICAL PROOF CONSTRUCTION

In this section we describe the hypothetical proving actions for a theorem chosen from a set of notes used in our one-semester, 3-hour per week, mid-level undergraduate transition-to-proof course. In this course, the students present in class their proofs of theorems from the notes and receive substantial criticisms and advice. There is no textbook and there are no lectures. The notes include theorems about sets, functions, real analysis, and abstract algebra, as well as definitions and requests for examples. The hypothetical proof construction we will describe is for one of the more difficult theorems in our notes and occurs near the end of the algebra section: *Theorem: If S is a commutative semigroup with no proper ideals, then S is a group.* To date, after two iterations of the transition-to-proof course, only one student (of 34) has persisted in proving this theorem correctly (albeit over the summer holiday). Thus, one can conclude that constructing a proof of this theorem presents quite a challenge.

The relevant background information for this proof construction is quite small. A semigroup is a nonempty set S with an associative binary operation that we will write multiplicatively as xy for elements x and y of S. Associativity means that for all elements x, y, and z of S, $(xy)z = x(yz)$. S is commutative means that for all elements x and y of S, $xy = yx$. If A and B are subsets of S, we mean by AB the set of elements ab where a and b are elements of A and B respectively. In this setting, a nonempty subset I of S is an ideal of S provided SI is a subset of I. Such an ideal is called *proper* in S provided it is not all of S. In this commutative setting, S is a group if it has two additional properties. First, there must be an "identity" element e of S so that for any element s of S, $es = s$. Second, given any element s of S there must be an "inverse" element s' of S so that $s's = e$.

In constructing a proof of the above theorem, it is easy to see that if I is an ideal of S, one can conclude I is not proper, that is, $I = S$. What is not so easy is trying to construct an ideal that "looks" different from S, and what that might have to do with producing an identity element e of S and inverses, in order to prove that S is a group. Since there is nothing else to work on, one must persist in trying to find an ideal of S without any idea of whether, and how, that would be helpful. It turns out that some students rather quickly think that if s is any element of S then $S\{s\}$ (also written Ss) might be an ideal of S, and hence equal to S. Once the idea has been articulated, it is not so hard to prove that Ss is an ideal. But how might $Ss = S$ help in proving that S is a group? Nothing in the students' notes says anything about solving equations in semigroups. However, if t is also an element of S, the above set equation means that there must be an element *x* of *S* so that $xs = t$. That is, the equation $xs = t$ can always be solved for *x*. It turns out that one can use the solvability of this equation in several ways to collect information which, for many students, is of unknown utility. Nevertheless this information, once collected, can be organized to show the existence of an identity element and inverses in *S*. To do this requires both persistence and a willingness to obtain whatever results, in the form of equations, that one can without knowing whether those results will ultimately be helpful. (See the proof in the Appendix.)

We could have added two easily proved lemmas to our course notes that would have made the proof of the above theorem much easier for our students. However, the purpose of the course is to learn to construct a variety of hard, as well as easy, proofs, and having relevant experiences is important in developing the students' ability to do so. We view learning to persist in "exploring" mathematical situations by obtaining "whatever one can get," even without knowing its ultimate usefulness, as an important part of developing students' proving abilities.

While the proof of the above theorem calls for persistence and exploration, proving in general can call on a whole "tool box" of knowledge and abilities, such as the use of proof by contradiction or mathematical induction, or looking for inspiration by proving easier theorems, perhaps by adding a hypothesis such as finite-dimensional or finite. However, discussion of such topics is beyond the scope of this paper.

A MATHEMATICIAN'S PROOF CONSTRUCTION

Our PhD student, Milos Savic (2012), investigated nine mathematics professors' proving using tablet PCs with screen capture software, as well as Livescribe pens and special paper, so that they could take the devices home and construct proofs in a naturalistic setting (without the time constraints and influences of an interview setting). All nine mathematicians' writing and speaking was recorded with time and date stamps. Several of the mathematicians acknowledged getting "stuck" on the above *Theorem 20: If S is a commutative semigroup with no proper ideals, then S is a group*, in a short set of notes containing only the material on semigroups. However, none gave up, as most students might, but persisted. One mathematician proved it the next day and another proved it after taking a break for lunch. Later, in a focus group interview, the professors indicated several ways they have of getting "unstuck" in their own research. These included getting up and walking around or doing something else for a while, as well as strengthening the hypotheses in order to prove an easier conjecture. It seems clear these mathematicians took the construction of the proofs in the semigroups notes as a positive challenge and had a sense of self-efficacy. Apparently this provided the motivation to persist, a crucial component of their success.

Several of the nine mathematicians volunteered that the material was both accessible and unfamiliar. However, they were unaware of the origin of the notes, that is, that they came from the course described above. Perhaps for this reason, several mathematicians attempted to construct counterexamples to some of the theorems. In attempting to prove Theorem 20, all nine mathematicians at some point considered "principal" ideals, a concept not in the notes, when considering the ideal *Ss*, where *s* is an element of *S*. This probably comes from remembering facts about ideals in rings; however, our students could not have had such memories as the course notes did not cover rings, and this course is a prerequisite for abstract algebra which would cover rings. Note that *Ss* = *S* is one of two key ideas in proving Theorem 20--ideas without which it is difficult to make progress.

Dr. G's Construction of a Proof of Theorem 20

Below we describe most of the work that one of the nine mathematicians, Dr. G, did when attempting to prove Theorem 20, which he eventually did successfully. Our description is taken from transcripts of Dr. G's speaking and writing while he worked alone using a Livescribe pen and special paper that recorded his writing and speaking with time and date stamps.

As seen below, Dr. G took a meandering path as he explored how to prove Theorem 20. His various "twists and turns" are indicated in bold typeface. Dr. G started at 7:02 a.m. by considering the statement of Theorem 20, but decided to **think about it** and have breakfast. At 8:07 a.m., he returned from a walk and **realized that gS** (where g is an element of *S*) **is an ideal**, so *gS* = *S*. He then **thought about inverses** and **struckthrough his entire previous argument**. At 8:09 a.m., he **noted that he needed an identity element** which is not given. At 9:44 a.m., he became suspicious that **Theorem 20 might not be true**, but noted that he had few examples which might show that.

At 9:48 a.m., Dr. G started **"tossing around" the idea** that a [commutative] semigroup with no proper ideals must have an identity, in which case, he could show it is a group. However, he didn't see why *S* should have an identity. He **began to think that translating by a fixed element** [an idea not in the notes] would move every element, which would mean there was no identity. Consequently, he **then began to look for a counterexample**. By 9:50 a.m. he neither saw how to prove Theorem 20 nor how to find a counterexample.

He then looked ahead to Question 22, the final task in the notes, which has three parts that ask whether certain semigroups are isomorphic. He **saw how to answer that and then looked at** *Theorem 21: A minimal idea of a commutative semigroup is a group.* He thought that he could probably prove that, so **he went back to Theorem 20**. By 9:51 a.m. Dr. G **recalled that he had earlier rejected Theorems 3, 9, and 12** of the notes and also did not believe that there are unique minimal ideals. By 9:53 a.m., he recalled that he had not been told any of the theorems were false and **looked at the non-negative integers** under multiplication. He saw that {0} is a minimal ideal and noted that the non-negative integers under multiplication do not form a group. He **thought that this was a counterexample** to Theorem 21, but had interpreted Theorem 21 incorrectly – something he later discovered and fixed.

At 9:54 a.m., he **started actually answering Question 22**. By 9:58 a.m., he had answered its three parts correctly. At 9:59 a.m. Dr. G. **took a break to think about Theorem 20** and at 10:08 a.m. he again attempted a proof of it. This time **he saw that for $a \varepsilon S$, there is $e \varepsilon S$ so that $ae = a$** and saw that e is "acting like …a right identity on a. Now why does it have to act that way on [an arbitrary] b?" By 10:12 a.m. **he found e' so that $be' = b$**, but that didn't help since he couldn't show that $e = e'$. Then at 10:13 a.m. he **saw that there is an f so that $b = af$**, and then by 10:14 a.m., he **saw that $be = afe = aef = af = b$**. At 10:15 a.m., he **saw that e is the identity** element. By 10:18 a.m., he had **used a similar technique to show S has inverses** and is thus a group.

Perhaps the most important thing about the above description of Dr. G's work is what is not there. There is no evidence that Dr. G thought there was anything wrong with having gone in all of those unhelpful directions or with having thought that some theorems were false, that he later discovered were true. What seemed to matter to him was the generation of ideas. If those ideas resulted in errors, one fixed them and learned from them. He exhibited persistence and a willingness to try argument directions that he clearly didn't know ahead of time would be helpful, and he altered directions when the need arose. It seems clear that Dr. G had the needed persistence, which was probably supported by a sense of self-efficacy with respect to his own mathematical research.

DISCUSSION

We are not the first to have considered the effect of affect and self-efficacy on mathematicians', and others', proving or problem-solving success. In their study of mathematicians' problem solving, Carlson and Bloom (2005) concluded that the mathematicians' effectiveness "appeared to stem from their ability to draw on a large reservoir of well-connected knowledge, heuristics, and facts, as well as their *ability to manage their emotional responses* [italics ours]." Also, in a study of non-routine problem solving, McLeod, Metzger, and Craviotto (1989) found that both experts (research mathematicians) and novices (undergraduates enrolled in tertiary-level mathematics courses), when given different experience appropriate problems, reported having similar intense emotional reactions such as frustration, aggravation, and disappointment, but the experts were better able to control them. This suggests that the mathematicians in the two studies had a mathematical self-efficacy belief that allowed them to persist.

TEACHING IMPLICATIONS

It seems to us that in order to do things that require persistence and exploration, a student is likely to need to believe that he or she can personally benefit from his or her persistence or exploration. That calls for a self-efficacy belief, which in turn calls for, perhaps numerous, successes in what is to be done. We see this as inherently difficult to arrange in traditional classes that generally involve students understanding a wide range of topics fairly quickly.

Below we will narrow our attention to learning to construct proofs in a transition-to-proof course, but we think something similar could be done in other situations that call for increasing self-efficacy beliefs about doing mathematics.

In order to maximize students' opportunities to experience successes in various aspects of proof construction in transition-to-proof courses, it would be good to have students constructing their own proofs as early as possible. However, this is often delayed in many such courses by an initial rather formal treatment of logic. While correct logic is essential for proof construction, we think its early and abstract treatment can be replaced by explanations of the relevant logic when needed in the context of the students' own work (in a "just-in-time" manner). This is because logic, beyond what most people know, actually occurs fairly rarely in student-constructed proofs (Savic, 2011).

While teaching logic can be integrated into discussing students' proofs and need not delay the start of their constructing proofs, there is an aspect of proving that is not usually explicitly taught early and that could be very helpful in facilitating student successes. There is a relationship between the structure of a proof, the logical structure of the theorem being proved, and the theorems and definitions used in constructing the proof. This relationship appears as part of the final written proof, but it can be isolated and considered first. One can write it first leaving blank spaces for the remaining work. We have come to call this a *proof framework* (Selden & Selden, 1995).

In constructing a proof framework, one is writing as much as possible of a proof before attempting to generate the ideas needed to complete it. One writes the hypotheses first, leaves a blank space for the body of the argument, and then writes the conclusion. Typically one then "unpacks" the meaning of the conclusion, and inserts the beginning and end of that part of the proof into the earlier blank space. One proceeds as far as possible in this way. This exposes the "real problem" to be solved. After the framework has been constructed, one can then try to generate the original ideas needed to complete a proof. Teaching students to build proof frameworks allows them to experience early successes. However, sometimes constructing a proof framework helps only a little in obtaining the final proof; this is the case for Theorem 20 discussed above. For other theorems, such as the theorem that states that the sum of two continuous real functions is continuous, constructing a proof framework can be very helpful (Selden & Selden, to appear).

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