APPROACHING ALGEBRAIC PROOF AT LOWER SECONDARY SCHOOL LEVEL: DEVELOPING AND TESTING AN ANALYTICAL TOOLKIT

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In this contribution, I discuss two episodes from a teaching experiment performed in lower secondary school (grade 7) aimed at introducing proof and using algebraic language as a proving tool. The teaching experiment was conceived within a theoretical framework presented in a PME Research Forum (Boero et al., 2010). That theoretical framework was further developed in order to improve the a-posteriori analysis and refinement of the classroom intervention. The aim of this paper is to show how the increased theoretical framework shed new light on the students' processes and also helped the design of further activities. More specifically, the analysis suggested occasions for developing argumentation at the meta-level.

INTRODUCTION

In this contribution I present and discuss two episodes from a teaching experiment, performed in grade 7, aimed at introducing a "proving culture" in the classroom. The contribution is situated in the stream of research outlined in a PME Research Forum (Boero et al., 2010). From a theoretical point of view, the Research Forum proposed an integration between Toulmin's model for argumentation and Habermas' theory of rationality (see the "Background" section below). The Research Forum paper ended with a series of suggestions for further developments and implementations, which were the starting point for the teaching experiment that is the object of this contribution. In the meantime, the retrospective analysis of some teaching experiments performed in the past (see Morselli & Boero, 2011) suggested to integrate the theoretical framework presented in Boero et al. (2010) in order to better frame the modelling activity of the student when he/she moves from a problem situation (internal or external to mathematics) to its algebraic treatment. In this paper I show how the integrated framework can be used to analyse the processes carried out by the students, with a special attention to the dialectic between proof and algebra.

BACKGROUND AND THEORETICAL FRAMEWORK

According to Balacheff (1982), the teaching of proofs and theorems should have the double aim of making students understand what is a proof, and learn to produce it. De Villiers (1990) suggests that the teaching of proof should make students aware of the different functions that proof has in mathematical activity: verification/conviction, explanation, systematization, discovery, communication. Stylianides (2007) proposes the following definition of proof that can be applied in the context of a classroom community at a given time:

> "Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: it uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification; it employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; an it is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community". (Stylianides, 2007, p. 291).

This definition brings to the fore that a smooth and meaningful approach to proof requires the students' progressive acquisition of basic content knowledge, but also the ability to manage (from a logical and linguistic point of view) the reasoning steps and their enchaining (modes of argumentation) and the ability to communicate the arguments in an understandable way. This is in line with the idea, exposed by Morselli & Boero (2009), that learning proof is approaching a specific form of rationality. The authors proposed an adaptation of Habermas' construct of rationality to the special case of proving, showing that the discursive practice of proving may be seen as made up of three interrelated components:

> "- an epistemic aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of "theorem" by Mariotti & al. (1997) as the system consisting of a statement, a proof, derived according to shared inference rules from axioms and other theorems, and a reference theory);
>
> - a teleological aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product;
>
> - a communicative aspect: the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture". (Morselli & Boero, 2009, p. 100)

Boero et al. (2010) proposed the integration of the construct of rational behaviour, with Toulmin's model of argumentation, thus creating a comprehensive frame that allows: 1) to better analyse students' proving processes; 2) to plan and carry out innovative classroom interventions. As regards the analysis of students' processes (1), the integrated model allows two levels of analysis: Toulmin's model focuses on the single argumentation step, while Habermas' construct allows to study each phase of the proving process, from the exploration to the final proof construction (thus shedding light on the legitimacy of reasoning steps, on the intentions behind each step, and on the communicational constraints). As regards classroom implementation (2), the integration suggests the importance of developing students' awareness of the constraints inherent in the proving process. Indeed, within the integrated frame, two levels of argumentation are outlined: the meta-level, concerning the awareness of the constraints related to the three components of rational behaviour in proving, and the level concerning the proof content. Within the integrated frame, students' enculturation into the culture of theorems is a long-term process where the teacher must create occasions for meta-level argumentations aimed at promoting students' awareness of the epistemic, teleological and communicative requirements of proving.

Crucial issues are: how to create occasions for meta-level argumentation and how to manage them in the proper way. Boero (2011) analysed a mathematical discussion at university level, showing that dealing at the same time with the content level and the meta-level is quite difficult, and suggesting some a-posteriori activities so as to create *occasions* for meta-level argumentation. In the present paper, I illustrate some *occasions* emerging from another teaching experiment, aimed at the approach to proof in arithmetic. Here the approach to proof is in a dialectical relationship with the introduction of algebraic language as a proving tool (i.e. the means to perform proof).

Algebraic proof

Boero (2001) describes algebraic treatment as a cycle: the starting situation (*sem1*) is put into formula (*form1*) by formalization. The first formula (*form1*) is transformed into another one (*form2*) that may give new information to the reader (thus performing an interpretation from *form2* to *sem2*).

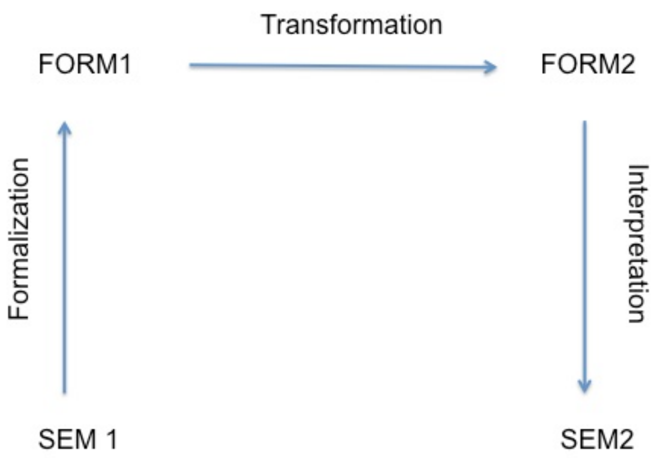

Figure 1: The cycle of algebra (Boero, 2001)

The fundamental cycle of *formalization*, *transformation* and *interpretation* is at the core of algebraic activity. In particular, algebraic proof is carried out by means of such cycles. When dealing with algebraic language as a proving tool, some crucial issues are: the choice of the formalism, that must be correct but also goal-oriented; the validity but also usefulness of the transformations; the correct and purposeful interpretation of algebraic expressions in a given context of use.

Morselli & Boero (2011), adapted the three components of a rational behavior in proving to the use of algebraic language in proving. In their elaboration, epistemic rationality consists of two distinct requirements: 1) *modeling requirements*, inherent in the correctness of algebraic formalizations and interpretation of algebraic expressions; 2) *systemic requirements*, inherent in the correctness of transformation (correct application of syntactic rules of transformation). Teleological rationality consists of the conscious choice and management of algebraic formalizations, transformations and interpretations that are useful to the aims of the activity. Communicative rationality consists of the adherence to the community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions.

The contribution of this paper relies in the integration of the fundamental cycle of Algebra with the construct of rationality (in the use of algebraic language in proving) in the analysis of students' behaviours.

RESEARCH PROBLEM

This paper concerns the experimentation of a task sequence aimed at approaching proof and introducing algebra as a proving tool. The main research questions are: can the analytic tool of rational behaviour integrated with the fundamental cycle of algebra be exploited to perform more in-depth analyses and interpretations of students' behaviours?

Previous research pointed out the importance of creating *occasions* for argumentation at meta-level, so as to make students aware of the epistemic, teleological and communicative constraints of proof. More specifically, with an eye to the use of algebraic language as a proving tool, it is important to promote reflection at meta-level on the nature of the actions to perform (*formalization*, *transformation*, *interpretation*). Thus, additional research questions are: are there *occasions* for meta-level argumentation? If yes, what are the themes for such an argumentation?

METHOD

The context

The teaching experiment is situated within the research project "Language and argumentation", started in 2008, aimed at the design, experimentation, analysis and refinement of task sequences for the development of students' "proving culture". Within the project, teachers and researchers share the same theoretical references and collaborate in the design activity, as well as in the analysis of the experimentation and the progressive refinement of the tasks.

The task sequences are conceived with argumentation as a core activity. Two types of argumentation are fostered: argumentation at content level, as a part of the proving process, and argumentation at meta-level, as a means for fostering reflection on the practices of mathematical proof related to the components of rationality. To this aim, tasks encompass: formulation of conjectures; comparison between different conjectures; justification of conjectures; comparison between individual processes and between individual final products. Didactical methodologies such as group work and mathematical discussions (Bartolini Bussi, 1996) are widely used. The team also explored the importance of making students to analyse students' written individual solutions, as it is advocated within the theoretical framework of the fields of experience didactics (Boero & Douek, 2008).

The task

The task sequence "*Sum of consecutive numbers*" was conceived for grade 7 (students' age: 13-14); it encompassed exploration, conjecturing and proving in arithmetic. The approach to proof is in a dialectical relationship with the introduction of algebraic language as a proving tool. The students were at their second experience within the project. They had already experienced the task sequence "*Choose a number*". In that occasion, they had appreciated the power of algebraic language for representing generality and showing the structure of the problem (see Morselli & Boero, 2011).

The task sequence was experimented in two classes, by two teachers involved into the project. The author, a researcher in mathematics education, attended all the class sessions, acting as a participant observer. This means that she observed the class sessions, could provide further explanations, if required, during the individual and group work and could intervene in the discussion that involved all the students. She realized video recordings of the mathematical discussions and collected all the individual and group productions provided by the students.

The whole sequence lasted about 10 hours. A description of the whole task sequence, as well a comparison between the two classes, is beyond the scope of this contribution. Here we confine ourselves to the first 4 hours. The students were proposed a first task ("*What can you tell about the sum of three consecutive numbers?*"). We may note that in both classes, in line with previous experiences in arithmetic, the students interpreted the task as referring to the sum of three consecutive *natural* numbers. The fact of working with natural numbers was not discussed with the teacher. The students worked individually, shared their solutions in small groups and after compared all the group solutions within a mathematical discussion. In each class, the discussion was devoted to the comparison of the conjectures and justifications provided by the students. For the aim of the paper, I selected from each discussion the excerpt referring to the classroom discussion about *how to justify the property by means of algebra*. The description of each episode is followed by a first analysis. Afterwards, an overall discussion of the results is presented.

TWO EPISODES FROM THE TEACHING EXPERIMENT

Episode 1: Three proofs for the same property

The students from the first class worked individually and produced different conjectures. Although the norms established in the classroom require that any answer should be justified, only one student accompanied his conjecture with a justification. Elio claimed that "*the sum is a multiple of three*", performed three numerical examples (see figure 2 for the original production) and wrote down: "*Moreover, if the third number gives a unit to the first number, we have three equal numbers*".

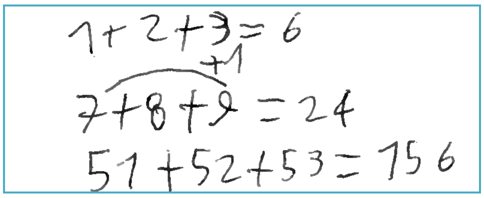

Figure 2: excerpt from Elio's individual solution

Elio's justification is firstly illustrated by means of a numerical example (7,8,9; see in figure 2 the line over the numbers, which represents the idea of 9 giving a unit to 7). This is a proof by generic example (Balacheff, 1982), since the numerical example is not aimed at "checking that the conjecture holds", rather to show "why the conjecture holds". The final sentence, although introduced by "moreover", is a justification in general terms. We may note that this proof by generic example has the function of *explanation*, not merely of *conviction*. This proof was shared by Elio to his group mates and, afterwards, presented to the whole class and discussed within a mathematical discussion. During the discussion, the observer and the teacher underlined that Elio's justification is a real *explanation* of the reasons why the conjecture holds. The observer also underlined that Elio's method shows that the property does not hold if the numbers are not consecutive, thus pointing to the function of *explanation*.

1 Observer: and in this way you understand why this is a property that not always holds. Some of you maybe tried to sum up three non-consecutive numbers. It is not sure that we still have this [*the divisibility by 3*], isn't it? This explains why we need three consecutive numbers to have it.

2 Elio: if we tried, here, instead of 503, with 504, I would get 503. I take away 1 [*from 504*] and I get 503, not 502.

3 Teacher: and you don't have anymore three equal numbers. The nice thing, using three consecutive numbers, is that if I take 1 away from the biggest number and I move it to the smallest number, I get three equal numbers. That why I always get three times the intermediate number, exactly because there is that "moving". [*they go on doing some numerical examples and applying the "taking away" strategy*]

Elio's individual solution contains also an algebraic proof:

a+a+1+a+2 could also be a+a+a+1+2

thanks to the commutative property it would be a*3+1+2

a*3+3.

During the discussion, Elio explained his choice of providing also an algebraic proof: "*But maybe they [numeric examples] did not work on great numbers and I could not do an example on all numbers*". We may observe that Elio was not completely satisfied with is proof by generic example, probably influenced by the common idea that "examples don't prove". Actually, this proof by generic example was already acceptable. We also observe that when passing from generic example to algebraic proof, Elio did not perform a translation in algebraic language of the same type of proof, rather he carried out a different proof. This fact was pointed out during the mathematical discussion. Elio, with the help of the teacher, created at the blackboard the algebraic version of his proof by generic example:

$$a+a+1+a+2=a+1 + a+1+ a+1=3(a+1)$$

In terms of cycle of algebra, the two algebraic proofs (the first one, carried out by Elio individually, and the second one, carried out during the discussion) may be modelled as it follows:

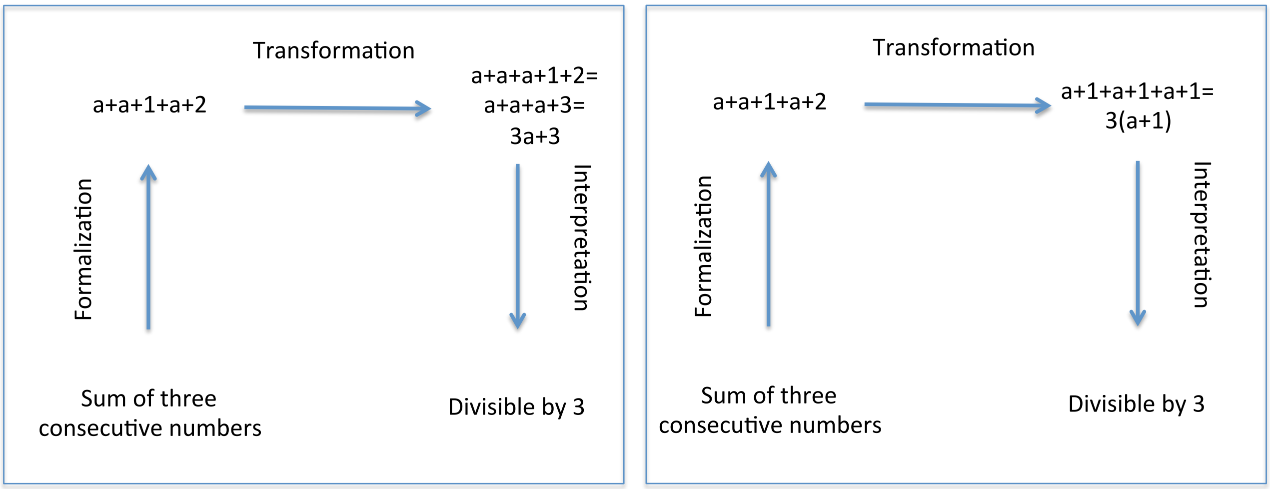

Figure 3. Elio's individual proof and the proof carried out during the discussion

Both proofs are carried out properly and each action (formalization, transformation and interpretation) involves some aspects of rationality: formalization is correct (modelling requirements of epistemic rationality), and useful, since it allows the subsequent treatment of the algebraic expression (teleological rationality). All the transformations are performed correctly (systemic requirements of epistemic rationality) and in a goal-oriented way, so as to obtain the divisibility by 3 (teleological rationality). In the second algebraic proof the strategy of "taking away" 1 is guided by the goal of getting three times the same number. Thus, transformation is even more goal-oriented (tel. rationality). Finally, in both cases interpretation is correct (systemic requirements of epistemic rationality), since the final algebraic expressions (3a+3 and 3(a+1) respectively) are read in terms of "divisibility by 3". In the first proof, 3a+3 could be more developed so as to make more evident the divisibility by 3. In the second proof the divisibility by 3 is evident and one could also note that the result is three times the intermediate number (epistemic rationality).

The analysis in terms of cycle of algebra and construct of rationality reveals some differences between the two proofs: the first one is mainly syntactical and could be carried out without having in mind the property to prove; on the contrary, the second one can be performed only under the guide of a strong anticipation (one must already have the goal of getting three times the same number); the second algebraic proof seems to be possible only in continuity with the argumentation in natural language and numerical examples (proof by generic example). Both proofs have an educational value and offer occasions for argumentation at meta-level. Indeed, the second proof is a telling example of proof as *explanation*, the first one may also convey the idea of algebraic proof as a means for *discovery*. Actually, also in second proof there is a discovery part, because also divisibility by the intermediate number turns to be evident. The second proof also highlights the importance of reflection on numbers. The analysis suggests that it would be important to promote an a posteriori comparison between them, thus fostering a meta-level argumentation on the way of carrying out algebraic proof (crucial role of *transformation*), and also on the value of algebraic proof (not only *conviction*, but also *explanation* and *discovery*).

Episode 2: struggling towards an algebraic proof

The same task was proposed in another class. One student (Edel) conjectured that "*the result is a multiple of 3*" and accompanied the conjecture by a first justification in natural language ("*because the summed numbers are three*") and by a symbolic expression (see figure 4). From the mathematical discussion, we know that Edel's intention was that of providing an algebraic proof for the property.

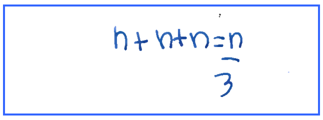

Figure 4. Excerpt from Edel's solution

In terms of cycle of algebra, Edel's attempt may be modelled in this way:

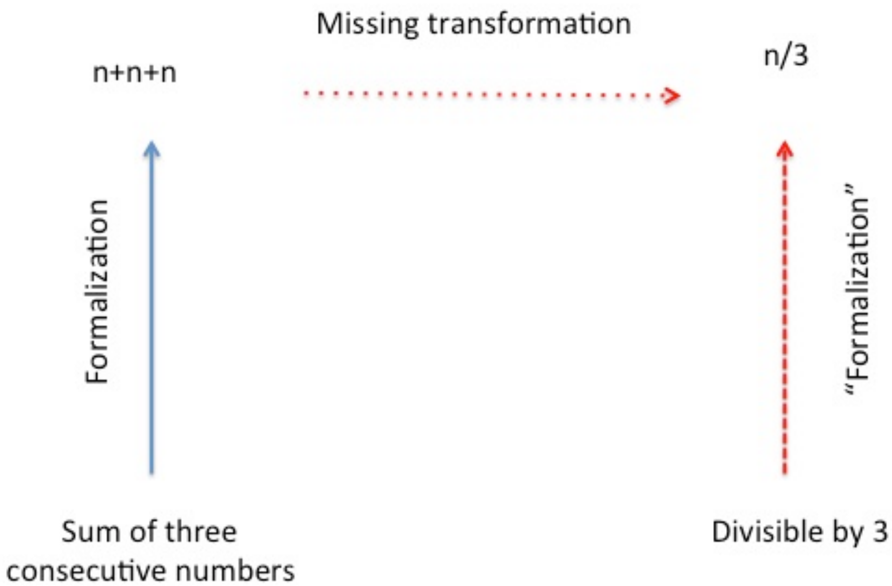

Figure 5: Edel's attempt

The cycle of algebra is not working in the proper way: formalization is not correct (since *n+n+n* is not a correct representation of three consecutive numbers) and *form2* (n/3) is obtained by an (uncorrect) formalization of the conjectured property "divisibility by 3" (*sem2*), rather than from a transformation of *form1*. From the point of view of rationality, we note lacks in the modelling requirements of epistemic rationality. Anyway, we suggest that the formalization of n/3 lacks in terms of epistemic rationality, but is rational from the teleological point of view: Edel wants to translate in letters what she already discovered, that is the divisibility by 3. We may say that the missing issue is exactly the *transformational* power of algebra. What makes algebra a powerful proving tool is the possibility of passing from the starting situation to the conclusions by means of transformation. This awareness (at meta-level) is completely absent in Edel's solution. Edel's activity has a teleological rationality, but according to her own goal: translating into letters. This is linked to her "ritual" conception of algebra as a proving tool: it seems that, for her, the algebraic proof is just a symbolic translation of what is already known.

Previous analysis suggests the necessity of a reflection on how algebraic language works as a proving tool (teleological aspects). We point out that there is a rationality in the choice of using algebraic language as a proving tool, and a rationality in performing the algebraic proof. Awareness of the teleological aspects referring to the use of algebraic language as a proving tool (it is a useful proving tool because it allows to obtain the proof by means of transformation of symbolic expressions) has direct consequences on the awareness of the teleological aspects referring to algebraic activity (formalization and transformation must be goal-oriented).

CONCLUSIONS AND FURTHER DEVELOPMENTS

We described and analysed two episodes from a teaching experiment aimed at introducing algebra as a proving tool.

The new integrated framework allowed us to put the requirements of epistemic and teleological rationality in a dynamic perspective. This brought to the fore that, when proving by means of algebraic language, the student must be able to combine the adherence to syntactical rules on one side, and the goal-oriented management of the processes of formalization, transformation and interpretation, on the other.

In this way, the integrated framework allowed us to understand better the students' processes (in particular, as concerns the nature of some of their difficulties) and to detect some occasions for argumentation at meta-level, that is occasions in which students can be asked to reflect on some aspects/components of the complex process they are involved in.

Important issues, to be treated at meta-level, concern: the role and value of numerical examples and the legitimacy of proof by generic example and proof in natural language; the crucial role of transformation, and the consequent importance of transformation-oriented formalizations; the dialectic between syntactic manipulation and more creative manipulation; and the possible links with the different functions of proof (from conviction, to explanation, to discovery).

The aforementioned task and the subsequent mathematical discussions can only partially achieve the goal of improving awareness on all these points. Further developments concern the design and implementation of tasks aimed at provoking such occasions.

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