NEW OBJECTIVES FOR THE NOTIONS OF LOGIC TEACHING IN HIGH SCHOOL IN FRANCE: A COMPLEX REQUEST FOR TEACHERS

Zoé MESNIL

Université Paris Diderot, Laboratoire de Didactique André Revuz

If everyone agrees that logic is needed to do mathematics, there are divergences concerning the role of mathematical logic in acquiring the necessary and sufficient knowledge in this area. Logic, often associated to reasoning, is also involved in language issues that are important in the research and writing of proofs. In France today, new goals are set for the teaching of notions of logic. A study of syllabuses and textbooks for high school in France shows strong constraints and ill-defined conditions for this teaching. There is therefore a need for training in logic, and it seems important to me to think about it in relation to the analysis of mathematical discourse.

INTRODUCTION

In the introduction to the proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education (2009), the authors note that some research should be pursued to understand the role of logic in the teaching of proof. The experience of teaching formal logic in high school during the time of "modern mathematics" in the 1970s in different countries has shown that this approach does not directly provide students with effective tools to improve their abilities in expression and reasoning. In France, logic was no longer part of the high school syllabus from 1981 to 2001. It was very soberly re-introduced in the 2001 syllabus. The 2009 syllabus, which is still in application, goes even further: it includes objectives for "mathematical notations and reasoning" which are linked to notions such as connectives AND/OR, negation, conditional propositions, equivalence, different types of reasoning. Behind those notions there are objects defined and studied by mathematical logic. But the aim stated by the syllabus is not to teach mathematical logic. Then, is it clear for teachers what the syllabus asks them to teach ? Did their initial training provide them with appropriate tools to carry out this teaching ? Some first observations made me doubt to be able to answer positively to these questions. First of all, the recent history of logic in mathematics teaching in France is a tormented history, and some teachers never got courses on notions of logic during their studies. Furthermore, since this teaching has been absent from syllabi these past years, teachers didn't really have to think about it during their training. Finally, a first glance on the activities proposed in the textbooks and on the content of the pages dedicated to the notions of logic evoked in the syllabus shows diversity in the interpretation of this syllabus, but also a certain lack of knowledge of the notions at work. This brought me to study in a more detailed way the system of conditions and constraints in which the teacher makes his teaching choices. Prior to this study, more general questions should be asked about the links between logic and mathematics and between logic and mathematics teaching. One of the aims of this study is to contribute to the reflection on the necessary training in logic for teachers, so they could teach the underlying logic in mathematical activity.

In this paper, I will mainly present this study. It shows the complexity of French high school teachers integrating notions of logic in their mathematics teaching. One axis of this work is to show the importance of language[1] issues in the contribution of logic to the mathematical activity. This dimension isn't taken into consideration in a precise way in the documents used as resources by teachers. Logic is often seen as linked to reasoning, in general and more specifically in mathematics. But it is also linked to the setting-up and the functioning of a language which allows to describe the structure of propositions and of reasoning, and the links between both. This is how I see mathematical logic as a reference theory. It deals with objects which could be tools to analyse our mathematical discourse and to understand the ambiguities inevitably linked to the use of certain informal formulations or implicit in the mathematical language.

First of all, I will present some quick thoughts on logic and language, and some didactical studies that have shown difficulties for some students probably linked to language. Later on, I will present the study of the system of conditions and constraints I have talked about. Finally, I will briefly describe the content choices made for a training called "Initiation to logic" for teachers in activity.

LOGIC AND LANGUAGE IN THE MATHEMATICAL ACTIVITY

The study of different moments in the history of logic shows that the constitution and description of a logic is accompanied by a necessary formalization of language, formalization in the sense of a codified formatting, the codes varying of course according to authors and eras. For example, in his logical work, *The Organon*, Aristotle (Greek philosopher, IV^{e} century BC) began to explain what he called a proposition (enunciative sentence, in which there is truth or error), and then classified these propositions according to two criterias, quality (affirmative or negative) and quantity (universal or particular). He obtained four types of propositions that are the building blocks of syllogisms, which could be treated in a formal way because the propositions can be replaced by variables. Nevertheless, Aristotle didn't want to formalize the relationship between the parts forming these propositions and didn't use logical constants like connectives and quantifiers.

Since the beginning, some mathematicians have sought the necessary and sufficient formalization of their language that ensures the infallibility of reasoning. For some of these mathematicians, the language had to be totally formalized, to reach the univocity of the meaning of expressions and the possibility of a formal treatment of these expressions, independently of their meaning. For another part of these

[1] The use of the term "language" does not refer here particularly to common language, neither to the mathematical one, but to the use by someone of signs he organizes with the intention of communicating something.

matematicians, too much formalization doesn't allow the intuitive progress of reasoning. Mathematical logic can be seen as the culmination of those researches. An important step has been crossed by modeling mathematical language and reasoning, allowing further exploration of the properties of these logical systems using mathematical tools.

These epistemological considerations lead me to the assumption that mathematical logic can provide tools, probably first to the mathematician and mathematics teacher, to analyze the language he uses in his mathematical activity, and to detect its ambiguities. For the teacher, an additional challenge is to provide tools for the analysis of students' reasoning, allowing to highlight another possible understanding. This can occur, for example with the propositions "if ... then ..." which are implicitly universally quantified but quantification is not always perceived by the students. That may lead them to give a response which is not expected, yet as the result of a correct reasoning. For example, some pupils could say that the sentence "if n is prime, then n is odd" is neither false, nor true, while almost all mathematicians say it is false (you can find a more detailed example with the "maze" task described in [Durand-Guerrier, 2004]).

In a more general way, various studies based on experiments with university students show the difficulties they have in understanding and proving quantified statements [Dubinsky, Yiparaki, 2000, Arsac, Durand-Guerrier, 2003, Chellougui, 2009, Roh, 2010]. For most students engaged in proving if a statement is true or false, the relationship between the quantified formulation of a statement and the framework of its proof is not clear. Thus, while recognizing the role of informal statements in memorizing mathematical results, J. Selden and A. Selden make the assumption that the ability to unpack the logic of an utterance by formally writing it is related to the ability to ensure the validity of a proof of this statement [Selden, Selden, 1995].

Unpacking the logic can be seen as writing the statements and making explicit some conventions, for example about quantification. The predicate language did not become the universal language in which the mathematicians express themselves, but a reference language, and, depending on the nature of their activities (research, drafting a communication, course ...), they use formulations whose logical structure is more or less exhibited. For example, calling a predicate P on a variable n which is a natural number, mathematicians commonly say "$P(n)$ when n is big enough." But to explain to students how to show this property, they may reformulate this as "$P(n)$ for all n beyond a certain value". And in a course, they could write "there exists N_0 such that for all $n \geq N_0$, $P(n)$". This interplay between different formulations, that is easy for mathematician, does not always seem so simple for students ! Though, I believe that reformulation work is important because it contributes to the construction of what J. Selden and A. Selden call "statement images" [Selden, Selden, 1995, p 133]:

> These are meant to include all of the alternative statements, examples, nonexamples, visualizations, properties, concepts, consequences, etc., that are associated with a statement.

This invites us to think about activities for students to develop the ability to reformulation, which is rarely an explicit goal of education and rarely proposed as an explicit task, and to think about the knowledge that teachers need in order to support such activities. S. Epp mentions the need for solid knowledge concerning language and logic [Epp, 1999]:

> When given by teachers with a solid command of mathematical language and logic, such feedback can be of enormous benefit to students' intellectual development.

HOW CAN A FRENCH HIGH SCHOOL MATHEMATICS TEACHER TEACH LOGIC TODAY ?

Research questions, study materials and methodology

There are some goals concerning "mathematical notations and reasoning" in the first high school year[2] mathematical syllabus, launched in September 2010. In these instructions, some objects of mathematical logic are explicitly mentioned. Therefore, teachers have to build up a teaching allowing to reach this aims. Doing so, they have choices to make, for which they are submitted to a system of institutional conditions and constraints. I think it is interesting to study this system in order to understand the teachers' practices concerning logic, and in order to think about the training they need for that. I have conducted this study following these three questions: what is the scholarly knowledge of reference for this teaching ?*("savoir savant"* [Chevallard, 1985]) ? What are the notions to teach (the knowledge to be taught, *"savoir à enseigner")* ? Where and why do we find logic in the high school mathematics teaching in France (ecological approach [Artaud, 1997]) ? I have looked for the answers by analysing different documents from 1960[3] till now. I think the historical approach is essential to understand the current choices.

The study of syllabi and their complementary instructions contributes to answer these three questions. I have noticed the notions of logic present in these documents, as well as the terms used to speak about logic. These terms give information about the specific function attributed to logic, in relation with language and reasoning. I have completed this study by searching in the APMEP[4] periodicals information on the reactions and expectations of the teachers, who are on the field of teaching. I have searched for the presence or not of articles on logic, and of debates concerning its teaching. Finally, I have studied what was said about logic in the textbooks. These textbooks are seen both as a possible interpretation of syllabus, proposing a "dressed knowledge" (*"savoir apprêté"*) [Ravel, 2003] and as a resource for teachers. I have

[2] 15-16 years old students.

[3] 1960 is the first year of the introduction of logic in the high school mathematics syllabi in France.

[4] Association des Professeurs de Mathématiques de l'Enseignement Publique, the most important association of mathematics teachers in France.

searched if and how the notions of variable, proposition, connectives and quantifiers were introduced, more specifically if and how their syntaxic and semantic aspects[5] were present and linked. I have also searched what kind of tasks were designed.

All these analysis show the complexity of the current demand : the conditions are ill-defined and the constraints are strong.

What is the scholarly knowledge of reference for the teaching of logic of mathematics ?

A part of this complexity lies in the logic itself. The question that interests us here is not the teaching of mathematical logic as a branch of mathematics. The question concerns rather the teaching of the logic of mathematics, which I defined as "the art of organizing one's speech in that discipline, seen under the double aspect of syntactic correction and semantic validity". One of the difficulties with this logic of mathematics is that there is no reference content, no consensus on knowledge in this area that are needed to do mathematics and on the words to use to say this knowledge. Here's an example of the potential problems with the lack of common knowledge reference. In the objectives of the new syllabus, students must "be trained on examples to properly use the logical connectives « and », « or » and to distinguish their meanings from the common meanings of « and », « or » in the usual language." But neither in the syllabus, nor in the accompanying document on this subject, entitled "Resources for high school first year, Notations and mathematical reasoning" can we find a definition of the logical connectives « and », « or ». However, the distinction between the use of these words in the language of mathematics and in everyday language is not limited to the inclusive or exclusive caracter of the connective « or », yet it is the only thing mentioned in the textbooks or in the accompanying document.

Another essential difference lies in the fact that in mathematical language, the connectives « and » « or » link two propositions, which is not always the case in the everyday language, even if they speak about mathematics. For instance, if we try to show the logical structure of the proposition "Sets A and B are not empty and disjoint", there will be three « and » that will correspond to connectives between propositions (in capital letters), and one that won't correspond: « Set A is not empty AND set B is not empty AND sets A and B are disjoint ». Not only is this distinction not evoked, but in addition to that, the study of certain exercises proposed in textbooks show confused conceptions about notions that yet seem as simple and usual in mathematics as the connectives « and » « or ». This shows that even if someone has studied mathematics and therefore used logic for this activity, it's still not enough for him to constitute a knowledge that could sustain a teaching of certain notions of logic to students. I make the hypothesis that the part of the mathematical logic which is generally called "propositional calculus and predicate calculus" is a corpus of

---

[5] In its syntactic aspect, a connective is an operator on propositions and its semantic aspect is given by the truth tables.

knowledge that can be a reference to teachers in their teaching of mathematics logic. The one condition for that is for it to be linked with the study of language and reasoning practices in mathematics. No such reference is explicitly proposed nowadays, on the contrary of what happened at the time of modern mathematics: in 1970, the instructions accompanying the new syllabus of the first high school year gave definitions and first properties of connectives and quantifiers.

What is the knowledge to be taught ?

Concepts of logic are mentioned for the first time in the mathematics syllabus for students in their first year of high school in 1960. During the middle of the twentieth century, the French mathematicians were strongly influenced by the Bourbaki group of mathematicians, whose axiomatic style spread into teaching. The reform called "modern mathematics" came into force in high school with the syllabus for first-year high school students in 1969. This syllabus was based on the idea of a unified mathematic, that could be used in experimental sciences as well as in human ones. Mastering the language of mathematics was then essential, and it was the essential function of logic. All textbooks at this time start with a first chapter on set theory and logic, which are the foundations of this language. The syntaxic and semantic aspects of the notions of logic were present. The instructions of 1970 specifically say that "the chapter *Language of sets* should become more of a practical introduction at any time in the course then of a dogmatic preamble", but they don't give examples of this practical introduction. The APMEP periodicals have published several articles between 1960 and 1975, linked to logic (theoretical presentations or stories of didactical experiences), and were spokesmen of animated debates such as the use of quantifiers symbols. This introduction of logic in the high school mathematics syllabi seemed then as an ambitious project, linked to the acquisition of mathematical language, and was sustained by a certain will to train teacher, not only to mathematical logic but also to modern mathematics in general. But if we look at textbooks, we can see that most of the time this logic has been reduced to simply a formal presentation of notions of logic, without linking it to mathematical activity. This presentation was not adapted to the initiation of a mass of student reaching high school to mathematics logic.

In 1981 came a new syllabus, radically different. It was the time of the "counter-reform", in which logic was explicitly excluded from mathematics teaching. Modern mathematics have been vehemently accused of being too formal, elitist and not linked to mathematics applications. Logic taught by then and some ostensive such as the tables of truth are almost symbols of this excessive formalism. This might be an explanation to the fact that none of the first high school year textbooks give tables of truth, even if some of them describe their contents, saying for instance that "the proposition **P and Q** is true only in the case where P and Q are both of them true".

This lasted until the implementation of the 2001 syllabus, which states that "training in logic is part of the requirements of high school classes". This text is included in the 2009 syllabus and is supplemented by a table setting targets for "notations and mathematical reasoning". These objectives relate to certain objects of mathematical logic, but are rather vague. It is also specified there that "the concepts and methods relevant of the mathematical logic shouldn't be the object of specific courses but should naturally take their place in all the syllabus chapters". I will give as examples two of these objectives: "Students are trained, based on examples, to correctly use the logical connectives « and » « or » and to distinguish their meanings from the common meanings of « and » « or » in the everyday language. They are also trained to wisely use universal and existential quantifiers (the symbols ∀, ∃ aren't due) and to spot implicit quantifications in certain propositions, particularly in the conditional ones." We have already seen that these objectives aren't clear, concerning the logical connectives « and » « or », and that the Resources document accompanying the syllabus doesn't take in charge all the important aspects of these notions. The second example concerns quantifications, mainly the implicit ones. We have evoked p3 certain didactical studies that have shown how these quantifications brought difficulties for the students, for instance in the comprehension of the propositions "if…when… ". Still, this doesn't appear in the Resources document. The syllabus instructions put logic both on the side of language and the reasoning one (presentation of the different reasoning types). But the link between logic and mathematics aren't the same as in the time of modern mathematics. Here, it's essentially about specifying the mathematical language's particularities related to the everyday language. Furthermore, the proposition, a base element of the mathematical language, is absent from syllabi and the Resources document.

This tormented history of logic in teaching mathematics in high school has especially two consequences that contribute to the complexity of what is actually asked from teachers. On one hand, all of them don't have the same training for these notions. On another hand, there hasn't been a continuous thinking, particularly a didactical one, about that teaching of logic notions in high school.

Furthermore, we have seen that logic should be taught along with other notions, which represents a strong constraint in matter of time. It should also be caught were it lies, which implies that teachers should be lucid and serene enough to be able to spot it.

Logic in the textbooks

For reason of space, I will only talk here about the actual textbooks. Textbooks authors should follow the directions, even those imprecise, given in the syllabus, to provide teachers with tasks for their students. Thus, in most textbooks published in September 2010, we can find pages offering a brief overview of the concepts of logic mentioned in the syllabus. These pages are not a separate chapter. They are a sort of glossary to which students and teachers can refer during certain tasks concerning logic. An analysis of ten mathematics textbooks published in 2010 shows a diversity in the presentations. Some books have one approach that can be called "propositional", which means they constitute a kind of "grammar of mathematical propositions". Other books have an approach that can be called "natural", which means that they take common language as the starting point for speaking about mathematical language, while specifying the requirements for this discipline, in particular the requirements of univocal meaning for each word. Because of the syllabus demanding that logic doesn't constitute a course of its own, very few textbooks use the term "definitions", "properties" in their pages, even though it's exactly what they give. Exercises associated to logic are essentially "True or False" ones, and don't allow to work on the language. Finally, we can find mistakes, such as the confusion between "if…then…" and "therefore", in the pages talking about logic or in the exercises of certain textbooks. The lack of knowledge of some teachers concerning logic makes it difficult for them to spot and analyse theses mistakes.

We have seen through the study of the documents contributing to define the *knowledge to be taught* that this knowledge isn't clear for the mathematics high school teacher today in France. It appears to us that it is therefore necessary to offer them trainings that will give them the needed tools for this teaching. Because logic is as essential to address language issues as it is to the reasoning ones in the mathematical activity, I think that mathematics teachers should have tools allowing them to think about the way they speak, and about the ambiguities and implicits that lie under some usual formulations in our practice of mathematics, and that mathematical logic seems a possible reference for this reflection.

AN INITIATION TO LOGIC IN THE FRAMEWORK OF A CONTINUOUS TRAINING FOR TEACHERS.

The Institut de Recherche pour l'Enseignement des Mathématiques (IREM)[6] at the Paris Diderot University proposed in 2011 a training course called "Introduction to logic" as part of the continuous training for teachers (this training course had already been organized in 2010 and renewed in 2012). This course was led in collaboration with René Cori, professor in the logic team of the Paris Diderot University. Fifty teachers (the number of places was limited) enrolled in this course, forty of them being effectively present. The training took place during three days of 6 hours each (two consecutive days in January, then one separated day a month later). One of the training goals was to give the trainees knowledge in mathematical logic. This doesn't mean lecturing about mathematical logic. It is all about teaching logic for teachers, a logic in context, at the service of mathematical activity. What is proposed is an analysis and a critical look on mathematical language with what teachers are already familiar. An important place is given to the notion of variable, that we will present as being characteristic of mathematical language in relation to the common language, and the multiple ways mathematics use to implicitly quantify their statements. The logical connectives are then presented as operators on propositions, which means they allow, starting from one or two propositions, to "create" a third one. This

[6] Research Institut for Mathematics Teaching

syntactic aspect is separated from the semantic aspect broached by giving the truth tables of these connectives. The notions of tautology and propositions logically equivalent are defined and put in relation with the practices of reasoning. An important moment was dedicated to implication: establishing its truth table creates reactions. Then, it is essential to note that the negative of a conditional proposition is not a conditional proposition. We also discussed a long time about the implicit universal quantification associated to the formulation "if… then…". We finally suggested few developments on the study of theories.

Another important part of the training was a more practical aspect, based on the study of the school textbooks. Basing themselves on selected parts, the trainees did a critical analysis in small groups. At the end, we shared our work. This practical exercise allowed to show the misunderstandings there can be about some notions, and was based on the taught theoretical components (this work was done after having spoken about variables, connectives and quantifiers). We also offered a moment during the third day for the trainees who wished to present activities they have done in class, so that we could discuss about it.

This training is a field for experimentation for my research. I'm trying to implement data collection and analysis tools.

CONCLUSION

With this contribution, I wanted to participate to the reflection on teaching notions of logic, particularly in showing the importance of not neglecting the contribution of the logic in problematics related to language and not just to reasoning. Anyway, language and reasoning are not two isolated poles, there are interaction between the structure of mathematical propositions and the structure of their proof. Moreover, reformulations are a common activity in the proving process in mathematics.

In France today the new syllabi for high school give goals concerning mathematical notations and reasoning, and mathematics teachers have then to explicitely teach some notions of logic. But through a study of documents defining the *knowledge to be taught*, we have shown that their choices for this teaching were in ill-defined conditions, and were subject to strong constraints. It seems then essential to me that the mathematics teachers have tools allowing them to think about the way they speak, and about the ambiguities and implicits that lie under some usual formulations in our practice of mathematics. Mathematical logic seems a possible reference for this reflection.

REFERENCES

Arsac, G., & Durand-Guerrier, V. (2003) Méthodes de raisonnement et leurs modélisations logiques. Spécificité de l'analyse. Quelles implications didactiques ? *Recherches en didactique des mathématiques. 23/ 3*, 295-342.

Artaud M. (1997) Introduction à l'approche écologique du didactique. L'écologie des organisations mathématiques et didactiques. *Actes de la IXe école d'été de didactique des mathématiques,* 101—139

Chellougui, F. (2009) L'utilisation des quantificateurs universels et existentiels en première année d'université, entre l'implicite et l'explicite. *Recherches en didactique des Mathématiques, 29/2,* 123-154.

Chevallard, Y. (1985) *La transposition didactique. Du savoir savant au savoir enseigné.* Grenoble, France : La Pensée Sauvage éditions.

Dubinsky, E., Yiparaki, O. (2000) On students understanding of AE and EA quantification. *Research in Collegiate Mathematics Education IV, CBMS Issues in Mathematics Education 8,* 239-289.

Durand-Guerrier, V. (2004) Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics, 53 (1),* 5-34.

Epp, S. (1999) The Language of Quantification in Mathematics Instruction. Chapter 16 in *Developing Mathematical reasoning in Grades K-12.* Lee V.Stiff and Frances R. Curcio, eds., Reston, VA: NCTM

Hanna, G., & de Villiers, M. (2012) *Proof and Proving in Mathematics Education. The 19th ICMI Study.* Springer.

Ravel, L. (2003) *Des programmes à la classe : étude de la transposition didactique interne. Exemple de l'arithmétique en Terminale S spécialité mathématique.* Thèse de l'Université Joseph Fourier Grenoble I.

Roh, K. (2010) An empirical study of students' understanding of a logical structure in the definition of the limit of a sequence via the -strip activity. *Educational Studies in Mathematics, 73,* 263-279.

Selden, A., & Selden, J. (1995) Unpacking the logic of mathematical statements. *Educational Studies in Mathematics 29,* 123-151.