MULTIPLE PROOFS AND IN-SERVICE TEACHERS' TRAINING

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This paper discusses a possibility to use interconnecting 'proof' problems that allow multiple solutions for teachers' professional development. Groups of teachers, which consisted of practitioners from various K-12 grade levels, were asked to produce several proofs of a given statement. I present a sampling of these proofs, which includes approaches and ways of reasoning specific for each grade level. Moments of teachers' collaboration and mutual influence are highlighted. This training method gives the participants an opportunity to make deeper mathematical connections as well as to understand better the culture of proof as a developing process along the entire mathematics curriculum across all grades.

INTRODUCTION

Proofs and logical explanations of mathematical ideas appear in various forms at all levels of mathematics education. Many mathematical facts can be observed and hypothesized at an early stage of a child development, perhaps in the primary or elementary school. They are introduced and explained by means appropriate for that level. However, the same facts may be proved later in the secondary school using more sophisticated vocabulary and advanced methodology, and illustrated at the university level using even more rigorous and abstract ideas. A mathematical problem, in particular, one that asks to prove a given statement, is called *interconnecting* if it obeys the following conditions: (1) allows a simple formulation; (2) allows various solutions at both elementary and advanced levels; (3) may be solved by various mathematical tools from different mathematical branches, which leads to finding multiple solutions, and (4) is used in different grades and courses and can be understood in various contexts (Kondratieva, 2011a, 2011b).

It had been proposed that a study of a progression of mathematical ideas that revolves around one *interconnecting problem* is useful for developing learners 'perception of mathematics as a consistent subject. Students familiar with the problem from their prior hands-on experience will use their intuition to support more elaborate techniques taught in the upper grades.

What is lacking about this approach, while it has a theoretical basis, is empirical evidence concerning teachers' practical reliance on it in their own classrooms. A related question is how to help teachers to adopt the interconnecting problem approach through their professional development. In this paper I discuss in-service teachers' experience with a problem that allows multiple proofs. A group of 25 mathematics teachers participated in this study in 2010 and another group of 20 teachers in 2012. Both groups took a graduate course on solving mathematical problems (I was the instructor) and this activity was one of the assignments in the course. Altogether nine subgroups of 5 students were formed, each of which included at least one primary/elementary, one junior high, and one senior high teacher. They were given one week to work on a problem stated below. Within each subgroup, the teachers were contributing solutions and explanations appropriate to the level they teach, reflecting on and checking each other's solutions. Each subgroup required to submit at least three distinct proofs. According to their responses, this activity allowed the teachers to view the statement from various perspectives and to see how it might be used within different grade levels and mathematical topics. In the next section I discuss a theoretical background of this research project. Then I analyse mathematical ideas that emerged from the teachers' collective work. In conclusion I comment on the possibility for teachers' development offered by this study.

THEORETICAL CONSIDERATIONS

Teachers' abilities to model proofs are very important for their students' progress in understanding of mathematics. Poor or confusing instruction produces a little learning (see e.g. Hsieh, 2005 as sited in Fou-Lai Lin et al, 2012). Thus, teachers' professional development in the area of proofs and argumentation is desirable (Stylianides & Stylianides, 2009). Teacher education should involve such classroom practices as spontaneous engagement in the processes of justification and evaluation of mathematical ideas and arguments (Simon & Blume, 1996). Solving proving tasks individually or sharing and criticizing each other ideas in small groups are recommended practices helping teachers developing their understanding of proof (Zaslavsky, 2005; Stylianides & Stylianides, 2009). In the process of proof construction, both self-convincing and persuasion of others are important (Harel and Sowder, 2007; Mason et al 1982). Teachers need to have an appropriate experience of proving processes in order to successfully implement them in their classrooms, which often would require a dramatic change in existing classroom practices (Douek, 2009). Such experiences include making conjectures, moving from experimental verification to general argumentation (Kunimune et al, 2009), becoming aware of limitations of current 'justification schemas' (Harel & Sowder, 1998), and developing more sophisticated or more efficient proofs. In particular, multiple proof tasks (Leikin, 2010) are found to be powerful "for guiding teaching and learning" (Sun & Chen, 2009) within a "spiral variation curriculum". It is also desirable that teachers were familiar with the culture of argumentation, made rational choices of mathematical tools and means of communication (Boero, 2011) and were explicit regarding transparency of proofs' presents in their classrooms discussions (Hemmi, 2008).

It had been observed that primary and secondary teachers might have distinct views on the appropriate ways and means of mathematical argumentation. This is generally consistent with cognitive stages of maturation of proof structures (Tall et al, 2012) as the learner grows from a child to adult. However, some elements of teachers' practices may cause an obstacle in their students' proper mathematical development. Teachers dealing with very young students use little symbolism and operate with 'quasi-real' mathematical objects (Wittman, 2009) and may be reluctant to accept other modes of argumentation. Elementary teachers also tend to rely of textbooks or more capable peers' information while constructing their own arguments (Simon & Blume, 1996). In contract, secondary teachers often reject verbal and visual proofs as being invalid (Biza, 2009) as they believe that all proofs must be formal algebraic (Dreyfus, 2009) and follow specific steps (Herbst, 2002). Consequently these teachers may focus on steps and algebraic details ignoring the overall logic of argumentation (Knuth, 2002). The idea that all proofs must be formal and rigorous leads some teachers to believe that proofs are inaccessible for grade school students, and thus excludes proofs from their pedagogical repertoire.

In this study, I was looking at a possibility to address several concerns and to adopt teachers' training recommendations found in the literature through teachers' engagement with interconnecting problems. When forming working subgroup of in-service teachers I specifically combined primary and secondary teachers in order to achieve the following outcomes: (1) expose elementary teachers to techniques and approaches employed in the secondary school; (2) expose secondary teachers to 'common logic' and intuition based explanations available at the elementary level; (3) let teachers to collaborate in solving an interconnecting proving problem, and thus let them to see and evaluate each other's concept of proof. Such collaboration could allow teachers to perceive proof as a developing and continuous process present in various forms at all stages of schooling. I was interested to collect an evidence of these processes as well as to find out what kind of assistance the teachers might need in order to benefit from solving interconnecting problems in mixed groups.

A PROBLEM AND COLLECTIVE POOL OF IDEAS

The problem for this study was chosen from (Totten, 2007) based on the criteria (1)-(4) for the interconnectivity listed in the Introduction.

Problem: Given a square ABCD with E the mid-point of the side CD, join B to E and drop a perpendicular from A to BE at F. Prove that the length of the segment DF is equal to the length of the side of the square.

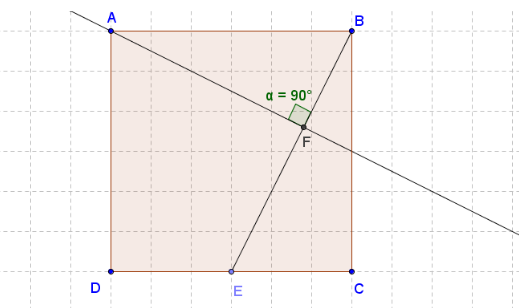


Figure 1: Problem to prove that length of DF is equal to the side of the square ABCD.

In this section I present all ideas generated by nine groups of five teachers. Some parts of original (given in italic) students' solution are summarized to save the space.

Approach 1. Direct measurement and comparison using various materials including a ruler, string, Popsicle sticks, compass, or dynamic geometry software.

> *I used point D as the center of a circle and placed my compass on point C which I knew was 6 units from point D, as seen in the diagram provided. I wondered if point F would be a point on the circle's circumference. I tried it, and sure enough point F was on the circumference of the circle. Point D was the center of the circle and both points C and A were on the circumference of the circle. Therefore, segment DF is the same length as line AD and line DC which are 6 units in length. As well it is the same length as line AB and line BC, since all four sides of a square are equal. Below* (see Figure 2, left) *is the graph with part of the circle described above drawn to show how I showed that segment DF was equal to the length of the side of the square ABCD:*

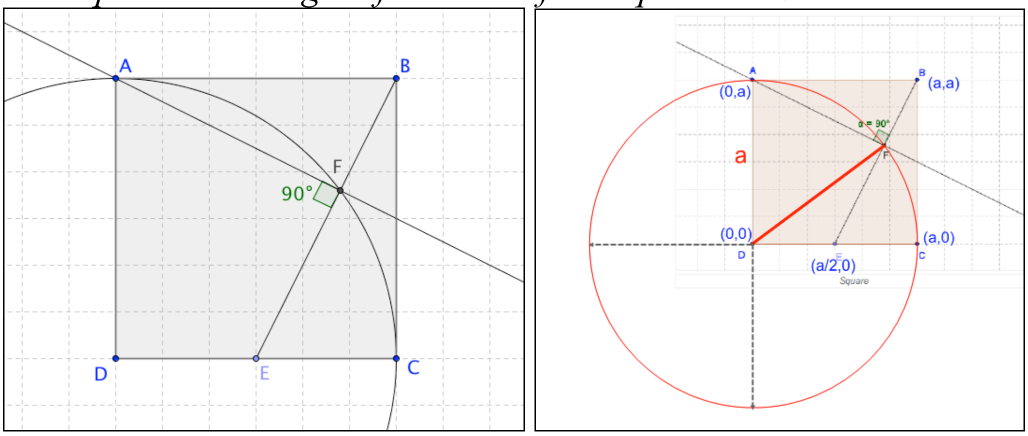


Figure 2: Approach 1 (use of compass) gives rise to the coordinate Approach 2.

This approach generated an algebraic method produced by another group member.

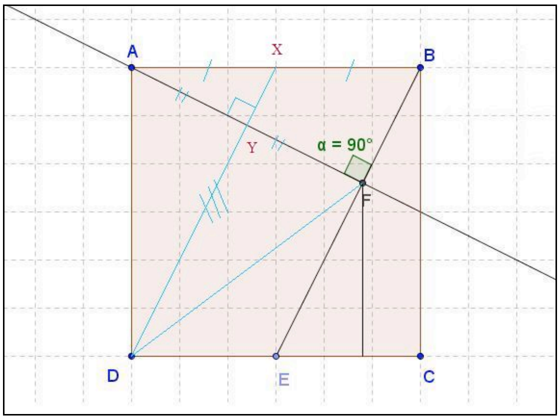
> *I* ***really*** *liked the idea that was suggested in Method 1 of drawing a circle through the points using D as the Center, but I am not sure how to generalize that. STUCK! Then I thought that if I could find the point F, I could sub it into the equation for my circle using a side length of "a". I know how to find F by using the equation for the line that intersects to make F. I can find the slopes using "a" as my side length, but how can I find the y intercept to finish my linear equations? STUCK! If I incorporate my previous idea of using points into this it will work! AHA!*

Approach 2. Use of Cartesian coordinates of the points (see Figure 2, right). Let's assume that D is located at the origin. Denoting the side length of the square as a we have the points $D(0,0)$; $A(0, a)$; $B(a, a)$; $C(a,0)$. To determine the coordinates of point F we must first find the equations of the lines BE and AF. We have two points $B(a, a)$ and $E(a/2,0)$ from which we may determine slope of the line BE and ultimately the y-intercept: $y=2x-a$; Since BE is perpendicular to AF, the slope of line AF is -0.5 and the y-intercept is a. Thus AF has equation $y=-0.5x+a$. The intersection point F is found by solving the equation $2x-a= -0.5x+a$, which implies $x=0.8a$ and $y=0.6a$. The distance DF between F and the origin D is the square root from the sum of squares of the coordinates of F, which after simplifications gives *a*, the side length of the square. This completes the proof. In students' version of the proof it reads:

> *So point F is* $(4a/5, 3a/5)$ *. Then to see if it fell on the circle, we just need to plug it into the equation for our circle. The circle has a radius of "a" so we have our equation:* $x^2 + y^2 = a^2$*. Now we just substitute our point F and see if it works:* $(4a/5)^2 + (3a/5)^2 = (16a^2 + 9a^2)/25 = a^2$*. Thus, our point F must be on the circle. Therefore, segment DF must be the exact same length as AD and DC which are also radii of the same circle and are also the sides of the square ABCD!*

Approach 3. Recognition of similar and congruent triangles (see Figure 3, left).

> *I had to find yet another proof. I liked Method 2 but needed to show that AFD is an isosceles triangle by some other method... I used the properties of similar triangles and congruent triangles to show that* $\overline{DF} = \overline{DA} = 6$ *units. Draw a line from point D to the midpoint of* AB*. Call this midpoint X.* DX *is parallel to* EB *because they have the same slope, so* DX *must intersect* AF *at a* 90^o *angle. Call this intersection point Y. Consider* ΔAFB *and* ΔAYX*.* $\angle FAB = \angle YAX$ *(common angle);* $\angle AFB = \angle AYX = 90^o$ *angles. Therefore,* $\Delta AYX \cong \Delta AFB$*. This means that* $\frac{\overline{AF}}{\overline{AY}} = \frac{\overline{BF}}{\overline{YX}} = \frac{\overline{AB}}{\overline{AX}}$*. Since:* $\overline{AX} = \overline{XB} = 3$ *units,* $\overline{AB} = \overline{AX} + \overline{XB} = 6$ *units, and the ratio* $\frac{\overline{AF}}{\overline{AY}} = \frac{\overline{AB}}{\overline{AX}} = \frac{6}{3} = 2$ *units. We know that* $\overline{AB}$ *is twice as long as* $\overline{AX}$*, so* $\overline{AF}$ *must be twice as long as* $\overline{AY}$*. Therefore,* $\overline{AY} = \overline{YF}$*. Consider* ΔDAY *and* ΔDFY*. We know:* $\overline{AY} = \overline{YF}$ *and* $\angle AFB = \angle AYX$ *are* $90°$ *angles, and* $\overline{DY} = \overline{DY}$ *(common side). Therefore,* $\Delta DAY \cong \Delta DFY$ *because of the Side-Angle-Side congruence property. So* $\overline{DF} = \overline{DA} = 6$ *units. QED*



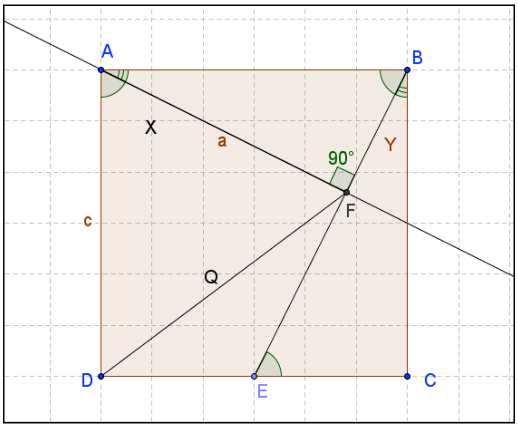


Figure 3: Approaches 3 and 4 look at triangle AFD in two different ways.

Approach 4. Use of trigonometry and the Cosine Theorem (refer to Figure 3, right).

> *I was looking at triangle AFD and thought that I could possibly apply the Cosine theorem to find the side of interest. Note that angles CEB, EBA and DAB are equal, call it X. Angles CBE and BAF are equal Y, and X+Y=90 degrees. Let the square has side c,* $\overline{AF} = a$ *and* $\overline{DF} = Q$*. Then we obtain the following.*

From right triangle ABF we have $c = a/\sin(X)$. From right triangle BCE we find $\cot(X) = 0.5$. From triangle AFD we conclude by Cosine Theorem that $Q^2 = c^2 + a^2 - 2ac\cos(X) = c^2 + a^2 - 2a^2\cot(X) = c^2 + a^2 - a^2 = c^2$. So, $Q = c$, or $\overline{AD} = \overline{DF}$.

Reflection: *Will this work for any size square? Yes because the ratio of the sides used in ΔBEC will always be 0.5 because E is the midpoint of side DC.*

Approach 5. Based on the recognition that AFED is cyclic (Figure 4, left).

*Aha! **ADEF** forms a quadrilateral and opposite angles $\angle ADE$ and $\angle AFE$ are both 90^o so they add up to 180^o. (We know $\angle AFE$ is a right angle because it's supplementary to $\angle AFB$**)** This means the other two opposite angles ($\angle DAF$ and $\angle DEF$) must also add up to 180^o since all angles in any quadrilateral add up to 360^o. Therefore, a circle can be constructed around the quadrilateral **ADEF** where each vertex, **A, D, E,** and **F** lie on the circle. Now it's likely I can prove it using arc measures.*

Now, note that angles CBE, BAF and DAE are equal, call this value *y*. Then $\angle DAF = 90^O - \angle BAF = 90^O - y$. From the relation for inscribed angles and arc measures we have $2\angle AFD = arc(ADE) - arc(DE) = 180 - 2y$. Thus $\angle AFD = 90^O - y$.

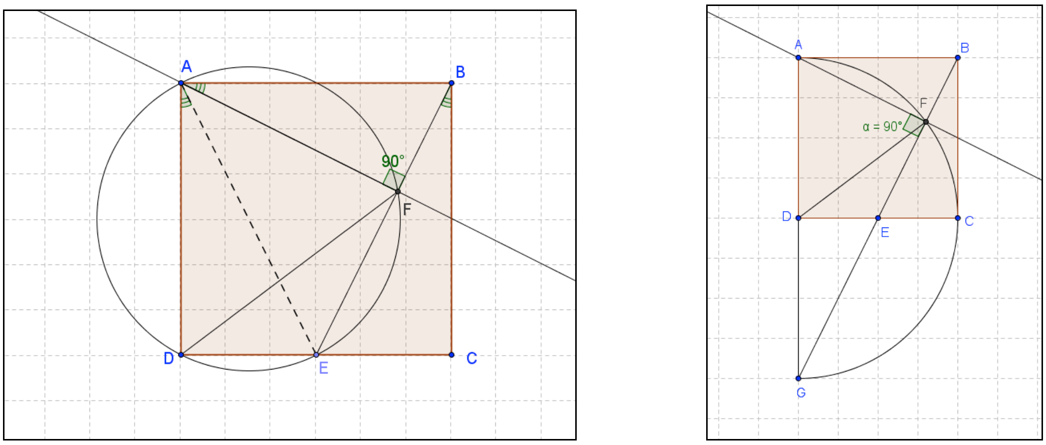


Figure 4: Approaches 5 and 6 use two different auxiliary circles.

*So line segments **AD** and **DF** must be of equal length since the isosceles triangle theorem states that sides which are opposite of equal angles in an isosceles triangle must also be equal. So **DF** is the same length as the sides of the square since **AD** is one of the sides*

ANALYSIS OF THE SOLUTIONS IN VIEW OF TEACHERS' TRAINING

Participants of this study were all enrolled in my graduate course on problem solving. Prior to solving the 'proof' problem presented in the previous section they practiced in solving several other problems individually and in groups. Their reading included the book by Mason et al (1982) regarding the stages of mathematical thinking and my paper (Kondratieva, 2011a) regarding the theory of interconnecting problems. Each group was asked to create as many as possible (but at least three) distinct solutions, individually comment on the thinking process highlighting AHA and STUCK moments (Mason, 1982), and as a group reflect on each others' approaches, identify their place in mathematics curriculum and select the most appropriate solutions for submission. Upon completion of this task teachers were asked to comment on the perceived usefulness of this training method.

Teachers' solutions and responses were examined in order to compare contributions from primary and secondary teachers as well as to observe the effect of their collaboration. Analysis of their work reveals the following.

First, while there was a disagreement about sufficiency of Approach 1 that involves direct measurement, almost all groups included this approach in their reports. Many primary school teachers provided detailed lesson plans on using various instruments helping students to construct and measure elements of the picture. Other group members often commented that in their view this approach is not qualified as a proof but still is very convincing and illustrative. This result concurs with literature stating that primary and secondary teachers may disagree about adequacy of some explanations. But, remarkably, this simple approach stimulated other group members to invent more rigorous justifications. An example of such collaboration is given in the previous section leading to development of Approach 2.

Second, majority of solutions dealt with concrete numbers. As it is evident from Approach 3, the teacher uses side of length 6 throughout her solution. While teachers had read about specialization and generalization techniques (Mason, 1982) and discussed them with their peers, still the tendency to use concrete numbers without further generalization was evident in the majority of papers. However, some teachers either made a comment on how to generalize their solution (see Reflection in Approach 4), and some had a proof in a general form (see Approaches 2 and 5).

Third, many teachers used approximate calculations. For example, in the trigonometric approach they would typically write "take $\arctan(0.5) = 63.4^{o}$", and then used approximate values of $\cos(63.4^{o})$ and $\sin(63.4^{o})$ to calculate the length of segment DF despite that the equality they were proving was exact.

Fourth, many submissions were very wordy and far from being mathematically efficient in reporting their final solutions. Even though the participants were asked to submit the best possible solution, many papers contained lengthy algebraic calculations that could be easily optimized. This likely reflects teachers' belief that every little detail must be brought up. But in doing that they often unnecessarily repeated or rephrased the same idea, and explained obvious things ("$\overline{DY} = \overline{DY}$ *common side*"), as can be seen e.g. in Approach 3.

Fifth, some groups submitted several solutions that employed the same mathematical idea and differed in very little details. It seems that the group members were hesitant to make their judgement and delegated the responsibility to choose the best solution to their instructor.

And finally, while the group members' collaboration was evident on several occasions and participants as a whole had produced a great deal of approaches, still there were solutions missed by the groups, even though elements of those solutions were present in the collective pool of generated ideas. As an example, the following approach was never proposed by the teachers, but when suggested by the instructor, they agreed that they were very close to discovering it by combining ideas from their Approaches 1, 2, and 5.

Approach 6. Extend lines AD and BE and call the intersection point G (Figure 4, right). Note that DE is the midline in ABG, that is points, E and D are midpoints of sides BG and AG respectively. Since AFG is a right triangle then its vertices lie on a circle, and hypotenuse coincides with the diameter of the circle. Thus $\overline{DF}$, $\overline{DC}$ and $\overline{DA}$ are all equal to the radius of the circle.

For completeness, I give another approach that employs a bit more advanced technique and can be used to illustrate the advantage of learning some further theorems in Euclidean geometry.

Approach 7. Based on Ptolemy's Theorem for cyclic quadrilaterals.

For cyclic quadrilateral AFED (Figure 4, left), the Ptolemy's theorem reads: $\overline{AF}\cdot\overline{DE}+\overline{AD}\cdot\overline{FE}=\overline{AE}\cdot\overline{DF}$. Let $\overline{DE}=a$. Then $\overline{AD}=2a$, $\overline{AE}=\overline{BE}=\sqrt{5}a$. Set up equations $\overline{BF}+\overline{FE}=\sqrt{5}a$, and $(2a)^2-(\overline{BF})^2=5a^2-(\overline{FE})^2=(\overline{AF})^2$ from right triangles AFB and AFE. Solving the system, we get $\overline{BF}=2a/\sqrt{5}$, $\overline{FE}=3a/\sqrt{5}$, $\overline{AF}=4a/\sqrt{5}$. Substituting these values in Ptolemy's equality we find $\overline{DF}=2a$, which is the side length of the square.

CONCLUSION.

This paper analyses a collection of proofs produced by groups of in-service mathematics teachers whose expertise ranged across all grade levels. Based on their responses, all participants of this study found it very informative to collaborate on one problem and produce proofs employing various methods and ideas capitalizing on the "interplay of empirical and theoretical argumentation" (Jahnke, 2008). In words of one teacher, "I never thought before of a possibility to prove the same claim in multiple ways. I was really amazed to see how many different approaches were proposed by my teacher-colleagues and how they all fit in different grades' math topics". This study reveals the potential of the use of interconnecting problems for teachers' training in mixed groups. Such setting allows teachers (1) to learn, evaluate, and criticize each other's solutions, (2) to share their ideas and to persuade their peers, (3) to collaborate on connecting intuitive and experimental methods with general argumentation, (4) to produce more efficient proofs, and (5) to choose appropriate tools and means to communicate their reasoning. Note that all these experiences are recommended in the literature for teachers' professional development. In addition, this training method allows teachers to see how different approaches are pertinent to different grades. Perceiving mathematics curriculum as a whole process of knowledge accumulation, teachers begin to acknowledge that many secondary school arguments are deeply rooted in primary/elementary level activities.

At the same time, some study participants did not fully benefit from the offered exercise. This suggests the necessity of more focused supervision and advising of mathematics teachers during their training. In particular, such advising should aim at developing habit of spontaneous moving from specialization to generalization, conscious distinction between exact and approximate calculations, and reviewing one's own solutions in order to eventually present them in a more general, insightful and concise form. The study also poses the questions: Why did all groups of teachers overlook certain approaches that clearly were within their capacity to produce? What can be done to ensure that teachers' collaboration realises the entire potential present in the individual contributions?

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