STUDENTS` USE OF VARIABLES AND EXAMPLES IN THEIR TRANSITION FROM GENERIC PROOF TO FORMAL PROOF

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First-year students are supposed to be able handle the deductive, axiomatic system of mathematics, to learn the formal symbolic language and to master different methods of proving. In this paper we report on our findings from a redesigned bridging course lecture for preservice teachers, in which the students were asked to construct generic proofs to complete their transition to formal proof, using their mathematical knowledge from school. The students` first assignment was collected and their handling and use of examples, generic proofs, formal proofs and variables were analysed.

Keywords Generic Proof – Example – Variable – Formal Proof

INTRODUCTION

The University of Paderborn offers a lecture specifically designed to help students to deal with higher mathematics. This course, "Introduction to the culture of mathematics", serves as a bridging course and was held for the first time in 2011/2012 as a requirement for the first year secondary (non grammar schools) preservice teachers. The course contents comprise logic, proof method, principle of induction, functions and sequences. Since the problems of the students with higher mathematics, especially proofs, are well known, the lecture`s main focus was on argumentation, refutation and proving. Yet, the mathematic was not presented in an axiomatic deductive system. On the contrary, the mathematics of the university were connected with the mathematics learned at school. In the context of proving, generic proofs were presented as a valid argumentation method for lower school grades, as a special tool for grasping the main idea of a proof and as a point of departure for formal proofs. So the generic proof was thought of as a didactic tool for enabling students to find the general argument and to fulfill a transition to formal proof by keeping the main idea and in addition using variables. However, after the correction of the first assignment, the students` work showed problems in their understanding of generic proofs and their use of variables.

Our goal in the current study is twofold: to investigate the students` problems with generic proofs and their use of examples in the proving process of statements and to document the obstacles in their use of variables as placeholders for concrete numbers in the field of elementary number theory.

THEORETICAL FRAMEWORK AND RELATED RESEARCH

Since Balacheff (1988) identified the generic proof as one of four main types in the cognitive development of proof, research on generic proofs (or generic examples) as a didactic tool for learning to prove has increased throughout the years (e.g. Leron & Zaslavsky, 2009; Mason & Pimm, 1984; Rowland, 1998 and 2002). But still Rowland is right, when he states: *[...] I am saying that the potential of the generic example as a didactic tool is virtually unrecognized and unexploited in the teaching of number theory, and I am urging a change in this state of affairs* (2002, 157). Also, the potential for teaching and learning to prove has not been exhausted yet: Firstly, generic proofs are said to be useful in order to convince students of the truth of a statement and *they enable students to engage with the main ideas of the complete proof in an intuitive and familiar context, temporarily suspending the formidable issues of full generality, formalism and symbolism* (Leron & Zaslavsky, 2009, 2-56). So secondly, they might pave the way for the transition to formal proof and help the students to handle variables and the structure of proof (e.g. Padberg, 1997). But in the transition from generic proof to formal proof two problematic areas arise: students` difficulty to generalize from the particular, to recognize and to take into account the generic quality of a generic example (Mason & Pimm, 1984; Nardi, 2008; Selden, 2012) and to adhere successfully to the mathematical language.

In this context the use of variables plays a subordinate role as they are a formal tool in the service of generalization. The research on students` use of variables has shown their difficulties with the concept of variables (e.g. Akgün & Özdemir, 2006; Cooper, Williams & Baturo, 1999; Ely & Adams, 2012; Philipp, 1992; Trigueros & Ursini, 2003). Although variables play various different roles in mathematics which cause various difficulties (Epp, 2011; Schoenfeld & Arcavi, 1988), their concept is rarely discussed in courses at university level. As Akgün and Özdemir (2006) explored in their case study, most students consider a variable in the context of equation as one specific number, even if it is used as a general number. In addition Trigueros and Ursini (2003) argued, that first-year undergraduates cannot distinguish between a variable as a specific unknown and a variable as a general number and that they have serious difficulties when variables are related to each other. They conclude: *Students` understanding of the concept of variable lacks the flexibility that is expected at this educational level (2003, 18).*

RESEARCH QUESTIONS

Our central research questions concern students` use of examples and variables in their transition from generic proof to formal proof:

1. How do first-year students argue when they are asked to construct “generic proofs”?
2. Do they also use the general argument found in the generic case in their formal proof?
3. What characterizes students` use of variables when formulating a formal statement and a formal proof?

THE LEARNING SEQUENCE

Before the students had to submit their first assignment, two lectures and one tutorial were given. In the first lecture, the aims of the first section ("Discovery and Proof in Arithmetic") were named: Getting to know the process of discovering and proving and distinguishing between verifications of a statement with concrete examples, with generic proofs and with formal proofs including variables. A research process was initiated by the question: "Someone claims: The sum of three consecutive natural numbers is always odd. Is this correct?". The statement was tested with some examples which led to the conjecture that the sum is always three times the mean number. As verification and explanation, a generic proof was presented (see table 1) and discussed by the students until the following statement was arrived at: "In this example, we are performing operations with concrete numbers, which are also possible with all (natural) numbers. Thus, this argumentation differs from our previous examples. It is a "generic proof", which includes a general argument. So here we have got a verification for the statement and an explanation, why the sum is always three times the mean number." Thus, "generic proof" was introduced by the lecturer and not invented by the students themselves.

Then, the general argument was used in the following formal proof (see table 2).

$$\begin{aligned} 1+2+3 &= (2-1)+2+(2+1) \\ &= 2+2+2 \\ &= 3\cdot 2 \\ 10+11+12 &= (11-1)+11+(11+1) \\ &= 11+11+11 \\ &= 3\cdot 11 \end{aligned}$$

TABLE 1: The generic example

$$\begin{aligned} k\in \mathbb{N}: (k-1)+k+(k+1) &= k+k+k \\ &= 3\cdot k \end{aligned}$$

TABLE 2: The following formal proof

After this, the "research process" in the lecture continued until the following surprising conjecture was fount: „The sum of k consecutive natural numbers is divisible by k, if and only if k is an odd number". The sum of k consecutive numbers with initial number n was defined as $S_{n,k}$ and finally the statement was formally proven.

In the tutorial groups, most of the time was needed to practice the representation of odd and even numbers by variables ($2n$ and $2n$-1, n є N), because it was needed in the first assignment, and the difference between an implication and a biconditional. In the following task in the tutorial group, the students had to use the representation of odd numbers as $2n$-1 to prove that a certain product is even. In the second task a proof by contradiction was needed, which one could obtain by using the formal representation of an even number.

TASK

The participants in the course were supposed to solve problems as a weekly assignment in order to get the permission to participate in the final test. The assignments of 64 students from four tutorial groups were scanned and their solutions for the following task were analyzed in an exploratory way to investigate the acceptance of the generic proof in a proof-oriented course and the students` use of variables. The task was as follows:

> *Prove the following statement with a generic proof and a formal proof. Before starting the formal proof, formulate the statement mathematically:*
> *The sum of an odd natural number and its double is always odd.*

TASK ANALYSIS AND EXPECTED SOLUTIONS

The generic proof

First, a generic proof consists of operations within concrete examples that can be generalized. Moreover, one has to find a generic argumentation, why the assumption is true in these specific examples. Afterwards one has to explain why this argumentation also fits all possible cases.

The generic proof (1):

$1 + 2 \cdot 1 = 3 \cdot 1 = 3$

$3 + 2 \cdot 3 = 3 \cdot 3 = 9$

$5 + 2 \cdot 5 = 3 \cdot 5 = 15$

Comparing the equations, one can recognize that the result must always be three times the initial number. Since three times an odd number is always odd, the result is an odd number.

The generic proof (2):

$1 + 2 \cdot 1 = 1 + 2 = 3$

$3 + 2 \cdot 3 = 3 + 6 = 9$

$5 + 2 \cdot 5 = 5 + 10 = 15$

Comparing the equations one can recognize that the second sum will always contain an odd and an even addend, because two times an odd number is always even. Since the sum of an odd and an even number is always odd, the result must be an odd number.

Formulating the statement mathematically

Formulate the statement mathematically: version (1):

$Let\ a\ \in N\ be\ an\ odd\ number. Then\ the\ sum\ a + 2a\ is\ also\ an\ odd\ number.$

Here the representation of an odd and an even number is not used, however, the use of one variable is necessary. This solution is in line with the expected level of knowledge of the students and the socio-mathematical norms created in the lecture.

Formulate the statement mathematically: version (2):
$For\ all\ n \in N\ there\ exists\ an\ m \ \in N\ with$:
$(2n-1)+2\cdot(2n-1)=2m-1.$

This is a more advanced solution, in which the use of two variables is necessary. Since n represents the first odd number, a second variable m is required. This statement also explicitly contains a universal statement and an existential quantifier.

The formal proof

Formal proof (1) - following the generic proof (1) and the statement (1):
$Let\ a\ be\ an\ odd\ number. Then\ a+2a=3a.$
$Since\ three\ times\ an\ odd\ number\ is\ always\ odd\ (*) the\ statement\ is\ proven.$

In this proof one can transfer the argumentation of the generic proof directly to the formal proof. The letter is used as a generalized number, as a generic element of a set of values. For the implication (*) we have to consider two possibilities: Either one can argue that the statement (*) is well-known in the sense of self-evident and true, which would be in line with the provided norms of the lecture, or one has to prove it, since it has not been proven before. In order to do this, one can argue: “3a = (a + a) + a”, “The sum of two odd numbers, a + a, is always even” and “The sum of an even and an odd number is always odd”.

Formal proof (2) - following the statement (2):
$For\ all\ n \ \in N: (2n-1)+2(2n-1)=6n-3=2(3n-1)=2m-1;$
$where\ m := 3n-1 \ \in N.$

Here the existential statement has to be shown for all $n \in N$.

RESULTS

The students` use of variables and their solutions for the generic proof, the formal proof and the formulation of the statement were categorized. But due to the size of this paper, we will just describe the categories for the generic proof and the formal proof in detail.

1. Types of students` proofs given as generic proofs

Students' proofs were classified into four different categories:

E0 The "generic proof" contains examples, which do not fit to the statement.

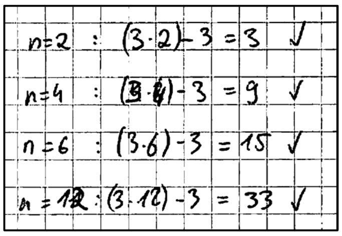

TABLE 3: a student proof, which belongs to the category E0

E1 The "generic proof" is just a verification by several examples without presenting the examples as generic. (These purely concrete examples are lacking explanations, further ideas or conclusions.)

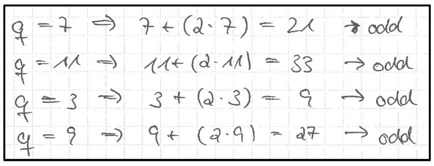

TABLE 4: a student proof, which belongs to the category E1

G1 The examples are presented as generic, but no further explanation is given.

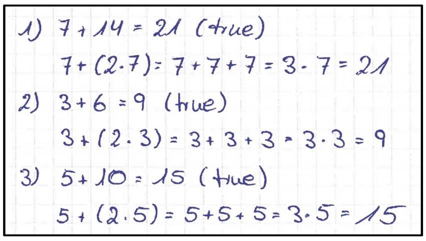

TABLE 5: a student proof, which belongs to the category G1

G2 The generic proof contains operations and ideas, which are named and generalized. (Here, the students identify different findings from operations, generalize them and use their findings in their argumentation process.)

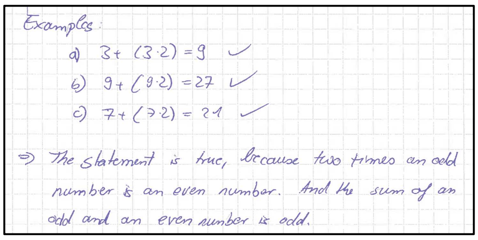

TABLE 6: a student proof, which belongs to the category G2

The frequencies of the categories are: E0: 3 (5.6 %), E1: 36 (67.9 %), G1: 8 (15.1 %) and G2: 6 (11.3 %).

So 39 Students (73.5 %) only presented examples in their "generic proofs" without connecting them with any argumentation to verify the statement generally (E0 + E1). Obviously, they have not understood the fundamental difference between a generic proof and verification by some examples. Out of the 14 students, who presented their examples as generic (G2 + G1), only six built an argumentation upon these in order to prove the statement in the generic proof (G1). In these six solutions, where the students succeeded in constructing the generic proof, the generic proof (1) was used two times and the generic proof (2) four times (see section "task analysis and expected solutions"). Four of these students did not use algebraic operations, but they argued verbally with the correct arguments.

2. Generic proof and formal proof

When the formal proof was successfully constructed, 18 students used formal proof (1), whereas none of the students used formal proof (2) (see section "task analysis and expected solutions").

Eleven students, out of the 14 belonging to category G1 and G2, tried to construct the formal proof and eight of these were using the same argumentation in the formal proof and in their previous generic proof.

3. Formulating the statement mathematically

34 out of the 64 students tried to formulate the statement mathematically including variables. Here the version (1) was used 21 times and the version (2) 6 times. 7 students used a mixed form of these. All of the students` statements included formal mistakes.

Moreover in version (2), the hidden existential statement was not made explicit at all and only one student explicitly mentioned the universal statement in the conjecture.

4. Types of formal proofs and formal mistakes with variables

Students' solutions of the formal proof were classified into four different categories:

P1 The reasoning in the formal proof is logical and correct.

P2 The reasoning process contains gaps and/ or statements are used that are not true in general.

P3 The reasoning does not contain any argumentation.

P4 Miscellaneous

(One student tried to prove a wrong statement. His solution is placed into the fourth category “Miscellaneous”.)

In most of the solutions belonging to category P2, the students did not give an explicit argument, why the term $3a$ or $3(2n-1)$ represents an odd number. We were expecting at least the argument "$3a$ (or $3(2n-1)$) is odd because both factors are odd”.

The quantitative results are shown in table 7:

Type of solution	Frequency	formally correct	with formal mistakes
P1	29 (51.8 %)	7 (12.5 %)	22 (39.3 %)
P2	18 (32.1 %)	1 (1.8 %)	17 (30.4 %)
P3	8 (14.3 %)	1 (1.8 %)	7 (12.5 %)
P4	1 (1.8 %)	0	1 (1.8 %)
Sum	56 (100 %)	9 (16.1 %)	47 (83.9 %)

TABLE 7: Frequencies of answer types

29 students (51.8 %) succeeded in constructing the formal proof (P1), of which only seven students accomplished this without formal mistakes concerning variables. 18 (32.1 %) students struggled with a correct logical argumentation (P2). In the formal proofs of another 8 students, no argumentation does occur (P3). In total, there are nine formal proofs (16.1 %) constructed formally correct and 47 (83.9 %) containing formal mistakes concerning variables.

In the process of proving, many students also struggle with the distinction between conditions, conjecture and proof. Sometimes the formulation of the statement is immediately followed by algebraic manipulations. Some students use different variables in the conditions and in the following proof.

The most common mistake in using variables is not clarifying to which domain the variable belongs. However this is essential in number theory. Usually numbers have to be whole numbers and not just rational numbers. Also, many students use $2n+1$ $n \in N$ as representation for an odd number, not considering that 1 cannot be represented hereby

When dealing with variables the students use a mixed form of everyday language and of the symbolic language of mathematics. Many students enrich the formal mathematical language with everyday language, when they seem to struggle with the formalism of the symbolic language. Letters and word symbols are used simultaneously, often without defining a correct domain. Moreover, one can recognize an inconsistent use of the symbolic language of mathematics. In addition, mathematical symbols like “=” or “϶” are often used in wrong ways.

FINAL REMARKS

The results of this case study are not representative, but they shed a new light on the current discussion about the role of generic proofs in the learning process of proving.

In our study, only a few students understood the idea of a generic example. This finding corresponds with the literature. It is well-known that preservice elementary teachers have difficulties in distinguishing proof and verification by examples (e.g. Martin & Harel, 1989; Recio & Godino, 2001). Furthermore, they have problems in understanding the explanatory power of generic proofs and in identifying the general idea in the particular case (Rowland, 2002). Yet, those students that recognized a common ground in the concrete examples were able to transfer it to the formal proof. In this transition to formal proof the students struggled with the formal language of mathematics, the use of the symbols and the meaning and definition of the variables.

Formulating the statement mathematically is another important part in the process of proving. At this point it becomes clear that this process requires more than just a correct argumentation in the formal proof. To prove a statement correctly, the initial statement must be understood with all its hidden universal and/-or existential statements. This is a valid starting point of a proper proof strategy.

One can consider different reasons, why the students in this study had such problems dealing with generic proofs. First of all, more time is needed to teach the idea of a generic proof in contrast to examples and formal proofs. Nevertheless, it was surprising that one lecture plus one tutorial devoted to the topic had such limited success. Also, tutors familiar with this example-oriented proof are needed to support the (didactical) ideas. Generic proofs had not been a topic in their previous mathematical lectures they had attended. Since it is well known that first-year students struggle with formal proofs and mathematical language in general, one has to be careful in using both examples and proofs in an argumentation process and one still has to consider the barrier that the formal mathematical language presents.

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