THE PRESENTATION AND SETTING UP OF A MODEL OF ANALYSIS FOR LEVELS OF PROOF IN MATHEMATICS LESSONS IN PRIMARY SCHOOL

Patrick GIBEL, Laboratoire LACES, Université de Bordeaux

Abstract: *The aim of this article is to establish the relevance of a model of analysis for the reasoning developed in didactical situations. We would like to describe our model stemming from theoretical frameworks, in order to highlight the elements which structure the analysis of the study of situations. We will analyse reasoning processes in a situation of validation involving a research dimension, proposed in primary school.*

Key words: reasoning, arithmetic, proof, argumentation, semiotics.

INTRODUCTION

The aim of this article is to present our model of analysis for reasoning processes (Bloch & Gibel, 2011) and to establish its relevance to analyse a sequence of mathematics, involving a research dimension, in a C.M.2 class (pupils age 10 to 11). The aim of the first part of the article is to describe briefly the model of analysis for reasoning processes elaborated by Bloch & Gibel (ibidem). In a second part, we will present te situation « The highest number », which is the subject of our study. We will make interest aspects and specificities of this engineering clear and propose an a priori analysis of this sequence. In the third part, we will use our model in order to identify and characterize the different forms of reasoning processes.

1. PRESENTATION OF THE MODEL OF ANALYSIS FOR REASONING PROCESSES

1.1 The theoretical tools used in the elaboration of the model

The subtle analysis of the reasoning processes produced, in a situation of validation, cannot be limited to an analysis in terms of propositional calculus based on Lorenzen's dialogical logics, as Durand-Guerrier points out (2007).

The need to take the semantic dimension into account, when analysing the reasoning processes, has contributed to sustain and justify our decision to take the Theory of Didactical Situations (TSD), elaborated by Brousseau (1998), as the foundation of our model. This theoretical framework must, however, be completed with the tools of local analysis, and with an analysis of the functions of the reasoning processes (Gibel, 2004) and of the signs, both formal and linguistic, which back it up. We will present the theoretical framework used to perform the semiotic analysis in the following paragraph.

1.2 The semiotic dimension of the analysis

During our previous research (Bloch & Gibel, ibid.) we underlined the fact that reasoning processes which appear in a classroom situation can take diverse forms: linguistic, calculative, scriptural, and graphic elements. Consequently the semiotic analysis constitutes one of the dimensions of our model completing those previously presented: on the one hand the function of the reasoning processes and on the other hand the corresponding level of the didactical milieu.

Pierce's semiotics seems particularly appropriate for our research and will indeed enable us to study more precisely the evolution and the transformations in the signs used by the different actors of the sequence.

In our application of Pierce's semiotics we will use the three designations: icon, index sign and argument-symbol. An iconic interpretation is based on intuition, sometimes based on a diagram, or using a programme of calculations; an index sign is to do with a proposition, for example, in the sequence studied, « the highest number is obtained by multiplying the five (whole) numbers given », considered as an argument-symbol is to do with a mathematical proof.

1.3 The didactical repertoire and the repertoire of representations

All semiotic means used by a teacher and those he expects from his pupils, through his teaching, form the didactical repertoire of the class as defined by Gibel (2004).

The didactical repertoire of the class can be identified as being part of the mathematical knowledge that the teacher has chosen to explain, namely for validation and during institutionalization.

The repertoire of representations is a constituent part of a didactical repertoire. It is made up of signs, diagrams, symbols and shapes and also linguistic elements (oral and/or written sentences), which make it possible to name the objects encountered and to formulate properties and results.

1.4 Methodology. The model of analysis for reasoning processes

The model of structuration of the didactical milieu used in this model, is that of Bloch (2006). The chart below (Table 1) sums up the levels of milieu – from M1 to M-3 – corresponding to the *experimental* situation.

The negative levels are of particular interest in the sequence studied i.e. the appearance of a proof process in the setting up of a situation having an adidactical dimension.

It is in terms of the articulation between the objective milieu and the reference milieu that we hope to see the expected reasoning processes appear and develop.

M1 Didactical milieu	E1: E-The universal subject	P1: P-planner	S1: Project situation	
M0 Learning milieu : Institutionalization	E0: The generic student	P0: The teacher teaching	S0: Didactical situation	Didac-tical situation
M-1 Reference milieu: Formulation & validation situations	E-1: E-The subject as learner	P-1: P Regulator	S-1: Learning situation	Adidactical situation
M-2 Objective milieu: action Heuristic milieu	E-2: E-The subject as an actor	P-2: P The teacher devolves and observes	S-2: Situation of reference	
M-3 Material milieu	E-3: E-The epistemological subject		S-3: Objective situation	

Table 1 *–Structuration of the didactical milieu*

In the previous research (Bloch & Gibel, 2011), we decided to focus on didactical analysis on three main "axes" of study used to guide our analysis of the reasoning processes.

The first axis of study is linked to the structuration of the didactical milieu: in a situation comprising an adidactical dimension, the pupils produce reasoning processes which depend to a great extent on the level of the corresponding milieu.

The second axis of study is the analysis of the functions of reasoning, which is, as shown above.

We will try to bring together the previous two axes of study by showing how the reasoning functions are linked to the levels of milieu and how these functions also *manifest* these levels of milieu.

The third axis of study is that of observable signs and representations. These things can be observed in different forms which affect the way the situation unfolds.

The application of the model to the situation will be following by an analysis of the milieu and semiotic analysis of the students and teacher's productions.

In conclusion, we classify reasoning, calculations, formulas, depending on the characteristics of the situation and knowledge(s) expressed by students in the previous phase, which reflect the level of milieu in which they are located, and the nature of signs produced. Our project consists in using our model to analyse reasoning processes in a situation of validation involving a research dimension, proposed in primary school.

2. PRESENTATION AND A PRIORI ANALYSIS OF THE SITUATION UNDER STUDY

2.1 Origin and interest in this sequence

The mathematics problem below was originally proposed by G. Glaeser (1999):

> Take any five natural numbers a,b,c,d,e. What is the highest number that can be obtained using the four elementary operations {+; -; x; ÷} applied to these numbers which can only be used once in the calculation; the same operation can, however, be used several times.

The setting up of this sequence at the C.O.R.E.M. is linked to the encounter of Brousseau and Glaeser, which was at the origin of this didactical project. The problem proposed is an open problem; G. Brousseau's idea is to get the pupils to discuss mathematical statements following rules which lead them to produce proof, and, more precisely, to find counter-examples.

The situation of validation, as conceived by G. Brousseau, is an adidactical situation or at least adidactical to a great extent: indeed it is possible and even probable that the teacher will have to intervene in order to ensure that the pupils carry on their discussion. Thus the analysis of this sequence, using our model, should enable us to provide some answers to the following questions:

What are the different forms of reasoning processes which come into play in the different phases of this sequence? What functions do they cover? What level(s) of milieu do they refer to? How does the repertoire of representations evolve during the sequence ?

2.2. ELEMENTS OF A PRIORI ANALYSIS OF THE SEQUENCE

2.2.1. Analysis of the nature of the expected answer to the problem proposed

To determine the nature of the answer expected by the teacher it is necessary to distinguish the conditions in which the answer must be given:

- If the sequence of numbers is given by the teacher, the answer expected is a number together with the programme of calculations enabling it to be obtained.
- If the five numbers are not given, that is to say if one is presented with a general case, then the proper answer will be a method of calculation. However, it must be underlined that writing an algebraic expression will not be appropriate because it is necessary to distinguish different cases according to the sequence of numbers under consideration.

  In the second case we are led to consider the algebraic expression

$$a \times b \times c \times d \times e$$

where a,b,c,d and e designate any five whole natural numbers

Yet this algebraic expression is only valid, to obtain the highest number, from a given 5-uplet, if none of the five numbers is 0 or 1.

The field of validity of this natural algorithm is not immediately obvious, it should lead the pupils to ask themselves questions about the status of the numbers 0 and 1.

It must be pointed out that, to obtain the highest number, it is necessary in the presence of one or several 0's to determine the highest number of the sequence of whole numbers excluding the zero(s), and to add the zero whole number(s) afterwards, which means distinguishing special cases when formulating the method.

2.2.2. Didactical analysis of the sequence

One of the objectives of this sequence is to make it possible for pupils to move progressively from arithmetic to general statements of methods enabling them to win. The sequence also aims to teach the rules of the game of proof: it is a lesson on right and wrong but also on the way of establishing it. The main objective of the lesson is therefore to put the pupils in a situation where they are led to discuss the validity of methods for obtaining the highest number, whatever the sequence of numbers proposed.

In this situation, it is anticipated that the rejection of a method will go together with the production of a counter-example, more precisely to the production of a sequence of five numbers and of a new method leading to the production of a new number – higher than the one obtained by the method proposed originally.

2.2.3. The unfolding of the sequence

The sequence is made up of three lessons; we are going to describe, for each of them, the different phases (cf. Annex).

3. ANALYSIS OF THE REASONING PROCESSES USING THE MODEL

In order to carry out an analysis of the reasoning processes produced, we use, first of all, the structure of the didactical milieu chart to distinguish the « embedded » situations corresponding to the different milieus (Brousseau et Gibel, 2005).

Concerning the sequence « The highest number », our model should enable us to analyse a posteriori, the transformations occurring during the reasoning processes produced by the pupils as regards their formulation, taking into account their functions in the didactical relation.

3.1 A priori analysis of the different milieu

The objective situation: the objective actor and the material milieu

The objective situation, the object of our study, is founded on the mathematical problem proposed. It is therefore a game situation for a given 5-uplet. The material milieu is made up of whole natural numbers. The required knowledge of the didactical repertoire that pupils will need are whole number operations and their properties.

The reference situation: the subject as an actor and the objective milieu

In the analysis of the sequence « The highest number », the teacher's objectives are, on the one hand, to devolve the situation of action to the pupils, that is to say to present the rules of the game, and, on the other hand, to lead the pupils to formulate the number obtained and the justification of the programme of calculation which goes with it.

The learning situation: the subject as a learner and the reference milieu

It must not be forgotten that the aim of the situation of formulation is that pupils write down a general method, that is to say one that can be used to obtain the highest number whatever the sequence of five numbers proposed.

So the objective is to get the pupils to produce a method whose field of validity is the largest possible. The phase of the formulation of methods aims at giving pupils the possibility to take a clear position concerning the action, and therefore to become fully aware of the decisions on which their actions are based, so that they can produce procedures whose validity can be discussed.

The didactical situation: the pupil and the learning situation

The level (M0) is that of assertions. At the previous level, (M-1), we were at the level of mathematical relations, the truth was obvious, the relation was right or wrong but there was no judgment made. Whereas at level (M0) the pupil, having the status of opponent in a situation of validation, arrives with a certain culture, and knowledge linked to the didactical repertoire he/she has at his/her disposal.

3.2 ANALYSIS OF THE REASONING PROCESSES

We will analyse two episodes indicating different phases of the lesson they are taken from. We use our model of analysis in order to study the different forms of reasoning processes which appear in the didactical relation, specifying their functions in relation to the associated level of milieu and indicating the signs) in reference with the didactical repertoire used.

Episode 1 : Discussion about the method lesson 2, phase 1 (cf Annexe)

A pupil : What method shall we use ?

Anne: You have to multiply all the numbers but when there are 1s you must add them.

Anne: Cédric you use Aline's method too, (that is to say – multiply all the numbers with one another in any order), and we will do another method to see who does best.

Anne: So 5,8,1,2,6 go on 5,8,1,2,6…5,8,1,2,6; so you two do Aline's method and you do my method; you have to find the highest number.

Tutor : Well try.

A pupil: OK, we're trying.

Anne: They are using Aline's method and we are using my method.

Anne: Have you understood ? You add the 1 to the highest number, to the highest number or any other number but you add it to a number. *(Her formulation suggests that the result is identical whatever number 1 is added to.)*

Nature and function – Milieu	Signs and repertoire
Level M-1. Reasoning linked to the formulation of a counter-example to Aline's method Proposition of a counter-example: formulation of a sequence of numbers, 5-uplet, used to compare methods and to produce a new method. Reasoning for the organization of tasks; distribution of roles among the group. Setting up the project. The sequence shown is in adequation with the project aiming at revealing the flaws of Aline's method: establishing the proof of the validity of Anne's method by experimentation in a certain case.	Local arguments aiming at revealing the flaws in Aline's method. The formulation of Anne's method is not precise. She did not indicate the number that 1 must be added to. Formulation of the purpose of the calculations i.e. the actual comparison of the results obtained by the two methods for a sequence in adequation with the project.

Anne: You add 1 to any number and then you multiply the lot.

Pupil 2: *I* wanted to put it last to add it up.

Anne: It doesn't matter, you can do it but I don't know if the result will be the same.(…) No, no it won't be the same.

Pupil 2: But it's the same thing!

Nature and function – Milieu	Signs and repertoire
Level M-2 Formulation of the method. The method is not precise as the number that 1 must be added to is not made clear. This interaction makes Anne aware of the fact that the results obtained differ depending on the number that 1 is added to.	Implicit use of the distributive property of multiplication compared to addition. Anne becomes aware of the influence of the number that 1 is added to on the result obtained. Decision about a mathematical object (Anne), concerning the influence of the number that 1 is added to on the result obtained

The rest of the discussion about the sequence 5,8,1,2,6

The experiment carried out by the group using the sequence proposed by Anne 5-8-1-2-6 shows that the result, (540), obtained using her method « adding 1 to the highest number », is higher than that obtained by Aline's method. (481). One can remark that the number obtained is indeed higher than the previous one but it is not, strictly speaking, the highest number possible.

The primary school teacher writes the two new methods proposed on the board:

Method 1 (Jérémy, Aline, Mélanie, Sylvain): The numbers are multiplied together in any order except for the 0 and the 1, which are added.

Method 2 (Anne, Sylvie, Cédric and Séverine): All the numbers are multiplied except when there is one 1 or several 1s, these are added to the highest number and they are all multiplied together afterwards.

CONCLUSION

The model underlines the reasoning processes produced and links them up to the previous knowledge and knowing of the didactical repertoire concerning elementary operations and the properties of multiplication. Concerning the formulations, the model shows their evolution in relation to the different milieu: one goes from giving sequences of arithmetic calculations to the formulation of general methods, of an almost algebraic nature, to end up with the production of semantic and syntactic arguments. This evolution takes place during a situation of validation. The semiotic analysis also shows the gap between the formalism introduced by the teacher and the repertoire of signs actually mobilised by the pupils throughout the sequence. This confirms the a priori analysis: the « final » situation being algebraic, it was difficult for the pupils at this level to attain a formulation of this nature.

REFERENCES

Bloch I. (2006) Quelques apports de la théorie des situations a la didactique des mathématiques dans l'enseignement secondaire et supérieur. Note de synthèse HDR de l'Université Paris 7.

Bloch I., Gibel P. (2011) Un modèle d'analyse des raisonnements dans les situations didactiques : étude des niveaux de preuves dans une situation d'enseignement de la notion de limite, Recherche en Didactique des Mathématiques, 31-2, 191-228, La Pensée Sauvage, Grenoble.

Brousseau G. (1998) Theory of didactical situations, Kluwer Academic Publishers

Durand-Guerrier V. (2007) Retour sur le schéma de la validation explicite dans la théorie des situations didactiques. In Rouchier A. (Ed.) Actes du colloque Didactiques : quelles références épistémologiques ? (Cédérom.). Bordeaux : IUFM d'Aquitaine.

Gibel P. (2004) Fonctions et statuts des différentes formes de raisonnement dans la relation didactique en classe de mathématiques, Doctorat de l'Université de Bordeaux 2, Bordeaux.

Gibel P. (2006) Raisonnement et argumentation : Analyse des différentes formes et fonctions des raisonnements des élèves en situation de débat à l'école primaire. In Bednarz N., Mary C. (Eds.) Actes du colloque EMF 2006 (Cédérom). Sherbrooke : Éditions du CRP.

Gibel P. (2008) Analyse en théorie des situations d'une séquence destinée à développer les pratiques du raisonnement en classe de mathématiques, Annales de didactique et de sciences cognitives, 13, 5-39, IREM de Strasbourg.

Glaeser G., (1999), Une introduction à la didactique expérimentale des mathématiques, Edition La Pensée Sauvage, Grenoble

Margolinas C. (1994) La structuration du milieu et ses apports dans l'analyse a posteriori des situations. In Margolinas C. (Ed.) Les débats de didactique des mathématiques, (pp.89–102). Grenoble : La Pensée Sauvage.

Peirce C. S. (1995) Le raisonnement et la logique des choses (Trad. par Chauviré C., Thibaud)

Annex Presentation of the sequence "The highest number"

The different phases of lesson 1

Phase 1: Devolution of the game. Sequence proposed 3,8,7,5,4

Phase 2: Complementary information.

Phase 3: Individual investigation

Phase 4: Pooling. Presentation of the results and designation of the winners.

Phase 5: Comparison of methods.

Phase 6: Instructions for the second game 7,3,2,5,8

Phase 7: Individual investigation.

Phase 8: Pooling. Presentation of the results and designation of the winners.

Phase 9: Instructions concerning the proposition contest.

Phase 10: Investigation

Phase 11: Regrouping. Formulation of the propositions. Discussion concerning the propositions.

Phase 12: Phase of the game 2,5,3,2,4

Phase 13: Presentation of the results.

The different phases of lesson 2

Phase 1: Instructions concerning the proposition contest.

Phase 2: Group investigation.

Phase 3: Pooling. Explanation of the results.

Phase 4: Discussion concerning the methods.

Phase 5: Phase of the game 5,2,4,0,3

Phase 6: Presentation of the results obtained using the methods.

Phase 7: Proposition of new methods.

Phase 8: Phase of the game 8,1,3,0,0

Phase 9: Presentation of the results obtained using the methods.

Phase 10: Proposition of a new method.

Phase 11: Looking for a counter-example.

Phase 12: Propositions of counter-examples. Discussion concerning the validity of the counter-examples.

Phase 13: Proposition of new methods.

Phase 14: Phase of the game (7-0-4-3-1).

Phase 15: Presentation of the results.

Phase 16: Looking for counter-examples.

Phase 17: Proposition of counter-examples.

The different phases of lesson 3

Phase 1: Pooling the results following the sequence proposed by Hélène (8-1-1-1-0)

Phase 2: Discussion concerning the status of Hélène's proposition.

Phase 3: Presentation of a sequence of numbers by the teacher (1-1-1-1-1).

Phase 4: Looking for the corresponding method.

Phase 5: Presentation of the methods. Explanation of the counter-example.

Phase 6: Phase of the game. (1-1-1-1-9)

Phase 7: Presentation of the results obtained using the methods.

Phase 8: Looking for counter-examples.

Phase 9: Phase of individual writing down a method.